ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

1. Professor D. N. Lehmer: *Bimagic squares.*

A bimagic square is one that is magic also in the squares of the elements. The author has proved the impossibility of bimagic squares of orders 3, 4, 5 and 6; and has also set up drastic restrictions on those of higher orders. (Received November 26, 1934.)

2. Dr. D. C. Duncan: *Completely symmetric elliptic self-dual curves of order* $4k$.

The well known formulas of Plücker are merely *necessary* conditions on the class, order, and number of singular elements of a plane algebraic curve, by no means insuring the existence of a locus, real or imaginary, having a designated set of consistent Plücker-elements. The present paper is a continuation of those appearing in this Bulletin (August, October, 1933; April, 1934), which establish existence of certain real self-dual loci of $p=0$, 1, whose singular elements are all real and distinct; such loci are impossible for $p>1$, since $k=n+2p-2$. The curves considered in this paper are very approximately realized by drawing $4k$ secant lines (omitting chords) through alternate points of $4k(k<1)$ points equally spaced about the unit circle. The equation in polar coordinates is $A\rho^{4k}\sin^2 2k\theta-(\rho^2-1)^3\prod_{i=1}^{2k-4}(\rho^2-a_i^2)^2=0$, in which the A, a_i, are the *unique* set of constants which cause the equation $A\rho^{4k}-(\rho^2-1)^3\cdot\prod_{i=1}^{2k-4}(\rho^2-a_i^2)^2=0$ to have $2k-4$ double roots. These loci are invariant under $8k$ collineations and $8k$ correlations, of which $4k+2$ are polarities. (Received November 3, 1934.)

3. Dr. D. C. Duncan: *The completely symmetric self-dual curve of order eight.*

In polar coordinates this locus has the equation $27\rho^8\sin^2 4\theta-256(\rho^2-1)^3=0$; its 8 cusps are equally spaced on the circle $\rho=1$, 8 crunodes equally spaced on $\rho=2$, 4 biflecnodes at infinity. The locus is autopolar by 8 rectangular hyperbolas and the 2 circles $\rho^2\pm(4/9)3^{1/2}=0$, which fact furnishes ready means for obtaining the bitangents and foci. The 64 foci lie by 8's on the 4 (double) cuspidal tangents $\theta=0$, $\pm\pi/4$, $+\pi/2$, corresponding to the 8 eighth roots of $(4/3)^8(1/12)$, and by 8's on the lines $\theta=\pm\pi/8$, $\pm3\pi/8$, at distances from the

origin corresponding to the 8 eighth roots of $(4/3)^8$ $(-1/12)$. The locus is realized very approximately by the configuration of 8 secant lines drawn through pairs of alternate points of 8 points equally spaced above a circle, omitting the chords within the circle. (Received November 3, 1934.)

4. Professor W. M. Whyburn: *Note on a theorem in abstract sets.*

The following theorem is established for certain abstract spaces which have the Borel-Lebesgue property: If G is a collection of closed point sets at least one of which is compact and if it is true that the sets of each finite sub-collection of G have a non-vacuous, connected set of points in common, then all of the sets of collection G have a non-vacuous, closed, connected set of points in common. Connectedness of the sets of G may be avoided by making G and the sub-collections mentioned in the hypotheses contain more than one element. Applications of the theorem are given in analysis. (Received November 5, 1934.)

5. Professor H. C. Hicks: *On the evaluation of the characteristic numbers of the Laplacean difference equation in rectangular regions.*

Certain characteristic numbers are used in the method of approximating to the solution of boundary value problems of partial difference equations introduced by L. F. Richardson. In this paper these characteristic numbers are evaluated for the Laplacean difference equation (first boundary value problem) in a linear region by means of the roots of a secular equation. The corresponding secular equation for a two-dimensional rectangular region is then solved by transforming it to an extended form of the equation for a one-dimensional region. Further extensions of this kind afford a determination of the characteristic numbers for an n-dimensional rectangular region. (Received November 22, 1934.)

6. Mr. Otto Laporte and Professor G. Y. Rainich: *Pseudo-minimal hypersurfaces in euclidean four-space.*

Ordinary minimal surfaces may be characterized by the vanishing either of the sum of the principal curvatures or of the sum of the principal radii of curvature. For hypersurfaces in four-space these two statements are not equivalent: reserving the word minimal for the first case we call pseudo-minimal those hypersurfaces for which the sum of radii vanishes. Using an integral representation generalizing Weierstrass' formulas developed by one of us previously, the problem reduces to a linear system of differential equations for the analogue of the second differential form. This system which takes the place of the Cauchy-Riemann system is reducible to a "Laplacian" equation (which is here of the third order) for a scalar potential. The principal solution obtained by imposing the condition of central symmetry furnishes a pseudo-minimal hypersurface produced by "revolving" a parabola around an axis. With a view of extending to the present case Maxwell's method of obtaining spherical harmonics by differentiation of the principal solution, we study the

group of transformations induced by four-dimensional rotations and find the infinitesimal transformations which applied to the principal solution furnish an infinite set of solutions. The investigation of the question whether a general solution may be developed into a series of such solutions is in progress. (Received November 23, 1934.)

7. Professor F. W. Perkins: *A set of completely independent postulates for differentiation.*

We consider a set C of real functions $f(x)$. To each function $f(x)$ in C, there corresponds a real function $\bar{f}(x)$, not necessarily in C. The functions $f(x)$ and $\bar{f}(x)$ are each defined for all real values of x. We now formulate the four following completely independent postulates: (I) The function $X(x)=x$ is a function in C; moreover there exist constants x_1 and q such that $\bar{X}(x_1)=1$, and $\bar{X}(x)$ is different from q for all x. (II) Given any constant x_2, and any function $f(x)$ in C, then $g(x)=f(x+x_2)$ is a function in C, and $\bar{g}(x)=\bar{f}(x+x_2)$. (III) There exists a constant x_3 such that if $f_1(x)$ and $f_2(x)$ are functions in C, and k is an arbitrary constant, then $F(x)=f_1(x)+k f_2(x)$ is a function in C, and $\bar{F}(x_3)=\bar{f}_1(x_3)+k\bar{f}_2(x_3)$. (IV) There exists a constant x_4 such that if C contains functions $f_1(x)$ and $f_2(x)$ for which $f_1(x_4)=f_2(x_4)=\bar{f}_1(x_4)=0$, and $\bar{f}_2(x_4)>0$, then corresponding to any such pair of functions there is a neighborhood of x_4 throughout which $f_1(x)\leqq|f_2(x)|$. *A necessary and sufficient condition that a function $f(x)$ have a derivative for every value of x is that $f(x)$ belong to some set C for which these postulates hold. If $f(x)$ belongs to any such set C, then the associated function $\bar{f}(x)$ is necessarily the derivative of $f(x)$.* (Received November 21, 1934.)

8. Dr. Hillel Poritsky: *Variation of characteristic parameter values (Eigenwerte).*

The variation in the parameter values λ considered is due to variation in the kernel K of a Fredholm integral equation $u(x)+\lambda\int K(x,s)u(s)ds=f(x)$. It is shown that if λ_0 is a non-degenerate characteristic value (that is, a simple root of $D(\lambda)=0$), then $\delta\lambda/\lambda_0=\iint\delta K(x,s)u_0(s)v_0(x)dxds/\iint K(x,s)u_0(s)v_0(x)dxds$, where δK is the variation in the kernel K, $u_0(x)$ the characteristic function (Eigenfunktion) corresponding to $\lambda=\lambda_0$, and $v_0(x)$ the similar function for the adjoint integral equation. For the differential system $L(u)+\lambda u=0$, $U_i(u)=0$, a more convenient form for the variation due to the variation δL in the differential expression L is given by $\delta\lambda=-\int v_0\delta L(u_0)dx/\int v_0u_0dx$. Variations are also considered due to change in boundary conditions or to varying boundaries. In particular, for the system $\nabla^2u+\lambda u=0$ in a region R, $u=0$ on the boundary S of R, there is obtained the formula $\delta\lambda=-\int(\partial u_0/\delta n)^2\partial nds/\int u_0^2dR$, where δn is the normal variation (displacement) of the boundary. (Received November 22, 1934.)

9. Mr. R. P. Bailey: *Convergence of sequences of positive linear functional operations.*

Let G denote the space of real bounded functions $\{x(t)\}$ of a real variable t defined over a finite interval (t_0, t_1), with the metric $(x, y)=\text{L.U.B}_{t_0\leqq t\leqq t_1}|x(t)-y(t)|$. A functional operation $U(x)$ defined over a subset E of G will be

said to be *positive* over E if $U(x) \geqq 0$, whenever x belongs to E and $x(t) \geqq 0$ in (t_0, t_1). The purpose of this paper is to give sufficient conditions for the convergence of certain sequences of positive linear functional operations defined over G or over vectorial subsets of G. Generalizations of convergence theorems due to Lebesgue, Fejér and others are obtained, and applications are made to mechanical quadratures, interpolation, and the singular integral theory. (Received November 22, 1934.)

10. Dr. G. B. Price: *Differential equations and expansion theorems in Banach spaces.*

This paper treats functions $f(x)$ of a numerical parameter x whose values are in a Banach space. The theory of the Riemann integral of such functions is developed, the results previously established being set forth and extended. The Riemann integral is applied in an extended treatment of differential equations, and also in establishing approximation and expansion theorems. (Received November 20, 1934.)

11. Dr. R. H. Cameron (National Research Fellow): *Linear differential equations with almost periodic coefficients.*

This paper deals with a linear homogeneous system E of differential equations with almost periodic coefficients having a module M. The object of the paper is to characterize all almost periodic solutions of E in terms of the module M. The result which is obtained is that the linear manifold of almost periodic solutions of E has a basis each element of which is the product of an exponential $e^{i\lambda t}$ and an almost periodic vector function whose module is contained in M. (Received November 21, 1934.)

12. Professor Morgan Ward: *Conditions for factorization in a set closed under a single operation.*

We call a set, closed under a commutative and associative operation, and such that not every element divides every other element, a band. By weakening the requirement for uniqueness in the decomposition of integral elements of the band into prime factors, a very general system is obtained giving as special instances the previous systems in which unique decomposition into prime factors occurs, studied by J. Koenig, F. Klein, A. H. Clifford, and others. (Received November 20, 1934.)

13. Dr. Saunders MacLane: *Some geometric aspects of abstract linear dependence.*

The notion of linear independence occurs frequently: the linear independence of the columns of a matrix, the independence of sets of arcs in a linear graph, the linear independence of vectors, etc. Whitney has defined a *matroid* as a system of elements with a suitably restricted concept of linear independence (this Bulletin, vol. 40, p. 663). In this paper, the relations between matroids and sets of points in a projective space are examined. This leads to the introduction of a geometric concept of duality for matroids. Finally, the fact that not every matroid is a matrix is shown to depend on the theorems of

Pascal and Desargues, together with certain of their generalizations. (Received November 21, 1934.)

14. Professor M. M. Culver: *On fundamental regions.*

We consider fundamental regions in S_{p-1} for the simple group of order $p(p^2-1)/2$ in $(p+1)/2$ variables, where p is a prime number greater than 3 and is congruent to 3 modulo 4. A set of $p+1$ Hermitian forms, which are permuted by the operators of the group, is selected. It is shown that there exist sets of values of the variables which give these Hermitian forms any desired arrangement when they are arranged in order of their magnitudes. The differences of these forms, when they are equated to zero and considered as hypersurfaces in S_{p-1}, divide S_{p-1} into $(p+1)!$ regions. (Received November 20, 1934.)

15. Professor T. R. Hollcroft: *The web of algebraic surfaces with a basis curve.*

A web of surfaces with basis curves has been treated heretofore only in the case of quadric webs with basis lines. In this paper are obtained the characteristics of a web of algebraic surfaces of order n with a basis curve which is of order μ_0 and class μ_1 and has only apparent double points. Associated with the web is an involution of points of order n^3-E, E being the equivalence of the basis curve on three surfaces of the web. The characteristics of this involution are obtained. The formulas are developed for one basis curve and then generalized for any number of basis curves of given orders and classes. (Received November 19, 1934.)

16. Mr. E. W. Cannon: *On semi-vectors and Lorentz transformations.*

Use is made of the fact that all proper orthogonal (real) four-rowed matrices may be written in the exponential form $O=\exp S$ (see Wedderburn, *Lectures on Matrices*, chapter 9), where S is a real alternating four-rowed matrix (=bi-vector), to factor O in the form O_1O_2 where O_1 and O_2 are *special* orthogonal matrices, the criterion for *specialty* being that the two vectors of the bi-vector S should be equal or opposite (furnishing respectively types O_1 and O_2). Any bi-vector S may be written as S_1+S_2 where $S_1=\frac{1}{2}(S+S^*)$ and $S_2=\frac{1}{2}(S-S^*)$ (where S^* is obtained from S by interchanging the two vectors of the bi-vector S) and $S_1S_2=S_2S_1$ so that $O=\exp S=\exp(S_1+S_2)=\exp S_1 \exp S_2=O_1O_2$. The special orthogonal matrices O_1 and O_2 furnish the apparatus for the description of semi-vectors (Einstein and Mayer) and a geometrical formulation was given by Clifford. The theory does *not* apply in the *real* field for Lorentz transformations where the role of S is played by SF, F being the matrix of the coefficients of the invariant quadratic form; in the complex field the results for Lorentz transformations follow from the theory for complex orthogonal matrices. (Received November 22, 1934.)

17. Dr. Robin Robinson: *Note on the geometric interpretation of the vanishing of a certain projective invariant of two conics.*

The envelope of a line which moves so that its intersections with two conics

form a harmonic set is a third conic, whose discriminant is a well known invariant of the pair of ternary quadratic forms representing the two conics. In case this discriminant vanishes, the eight tangents to the two conics at their four points of intersection pass by fours through two points. This paper establishes a number of other geometric properties, which characterize such a pair of conics, and to which the author finds no reference elsewhere. (Received November 21, 1934.)

18. Dr. J. C. Knipp: *Relation between the Fregier surface and the evolute.*

Let P be a fixed point on a quadric Q in 3-space, and let A_1, A_2, A_3 be points on Q such that PA_1, PA_2, PA_3 are mutually perpendicular. Then corresponding to any point P on the quadric surface there exists a unique point F, the Fregier point, which always lies in the variable plane $A_1A_2A_3$. The locus of F as P moves over the quadric is the Fregier surface. This paper considers the relation of this surface to the evolute. The two surfaces are found to be tangent along a certain curve and this curve is shown to be exceptional for the Fregier surface in several ways. (Received November 10, 1934.)

19. Dr. I. J. Schoenberg: *A new analogue of the Gauss-Bonnet formula with applications to geodesic triangles.*

Let Σ be an ordinary surface and a, b, c, A, B, C the sides and angles of a geodesic triangle of Σ. From classical results of Gauss (see G. Darboux, *Théorie des Surfaces*, vol. 3, 1894, p. 168) it follows that the quantities (1) $b^2+c^2-a^2-2bc\cos A$, $c^2+a^2-b^2-2ca\cos B$, $a^2+b^2-c^2-2ab\cos C$, and therefore also the quantities (2) $a-b\cos C-c\cos B$, $b-c\cos A-a\cos C$, $c-a\cos B-b\cos A$, have the sign of the curvature K of Σ within the triangle ABC, provided the triangle ABC is *sufficiently small*. Here it is proved that without any restriction on the size of ABC (1) and (2) are all negative for surfaces Σ of negative curvature. Moreover, these quantities are positive if K is positive, provided at least two of the vertices of ABC have the following property: The geodesics passing through one of these vertices form a field within the triangle and similarly for the other vertex. Examples show that these conditions are not superfluous. (Received November 24, 1934.)

20. Professor J. M. Clarkson: *An involutorial line transformation determined by a congruence of twisted cubic curves.*

Two pencils of quadric cones with the vertex of each pencil on every cone of the other pencil define a congruence of twisted cubic curves. An arbitrary line t of space is bisecant to just one cubic of the congruence and meets a fixed plane γ in a point T. Take a Cremona involutorial transformation in γ, and T is transformed thereby into T'. Through T' there exists one and only one line t' bisecant to the cubic of the congruence defined above. We shall say that t is transformed into t' by the *involutorial line transformation I*. The order, invariant lines, and singular lines of I are discussed. (Received November 16, 1934.)

21. Mr. L. B. Robinson: *Quaternions and inversive geometry.* Preliminary report.

Dr. Morley's recent book, *Inversive Geometry*, should revive interest in that branch of mathematics. Inversive geometry in four dimensions is a branch of quaternions. Dr. H. B. Phillips seems to have made the first steps in this direction, that is, putting inversive geometry into four dimensions. The author has extended his work from the hypersphere onward. For a hypersphere is a Cassinoid with one pole. The n-poled Cassinoid hypersurface can be written as the sum of products of n pairs of conjugate quaternions equated to a constant. The n-poled Cassinoid is a canonical form for more general hypersurfaces. So we can study many hypersurfaces by the use of *pairs of conjugate quaternions* as Dr. Morley in the plane uses pairs of conjugate imaginaries. (Received November, 24, 1934.)

22. Professor Morgan Ward: *A problem in recurring series.*

If a sequence of rational integers is defined recursively by a linear relation of fixed order, it may happen that all the terms of the sequence after a certain point are divisible by a fixed rational integer. We find here all such sequences, and determine in a very simple manner where the fixed divisor first appears, given only the initial integers of the sequence and the coefficients of the recursion relation. (Received November 20, 1934.)

23. Professor Morgan Ward: *A determination of all possible systems of strict implication.*

It is known that C. I. Lewis' postulates for his system of strict implication admit of three distinct interpretations. It is shown here that no other essentially distinct interpretations are possible. (Received November 20, 1934.)

24. Dr. J. B. Rosser and Dr. S. C. Kleene: *The inconsistency of certain formal logics.*

It is shown that certain formal logics are inconsistent in the sense that every formula in their notation is provable, irrespective of its meaning under the interpretation intended for the symbols. This results from the presence in these logics of a form of the Richard paradox, which is deduced by utilizing representations of the logics within themselves. In particular, the combinatory logic of Curry (American Journal of Mathematics, vol. 52, pp. 509–536, 789–834, vol. 54, pp. 551–558, and Annals of Mathematics, vol. 32, pp. 154–180, vol. 34, pp. 381–404, vol. 35, pp. 849–860) and a system of Church (Annals of Mathematics, vol. 33, pp. 346–366, vol. 34, pp. 839–864, and Kleene, ibid., vol. 35, pp. 529–544) are shown to be inconsistent. The inconsistency of the former results from its combinatory postulates together with the postulates governing material implication and the universal quantifier; the inconsistency of the latter results from the postulates governing functions and the properties of formal implication and the logical product. (Received November 16, 1934.)

25. Professor Richard Courant: *A general theorem on partial*

differential equations of second order with constant coefficients and some applications.

Any equation of this type can be reduced to an "ultrahyperbolic" equation for a function $u(x_1, x_2, x_3, \cdots, x_n; y_1, y_2, y_3, \cdots, y_n) = u(P, Q)$ of $2n$ independent variables x_i and y_i (or two points P and Q in an n-dimensional space) having the form $\Delta_x u = \Delta_y u$, where Δ_x, Δ_y denote the Laplacian operators with respect to the x_i and y_i, respectively. Let $\overline{P}$ and $\overline{Q}$ denote points at the distance r from P and Q respectively in the n-dimensional space. Then the mean-value theorem discovered by Asgeirrson states that the mean values of $u(P, \overline{Q})$ and $u(\overline{P}, Q)$ are equal, regularity of our function in appropriate domains assumed. From this theorem we can deduce many special results concerning potential theory, wave motion, and a more refined investigation of the initial value problem for non-elliptic equations in case the initial manifold has not the usual space character. (Received November 27, 1934.)

26. Professor H. T. Davis: *An extension of a theorem of Gauss to polygamma functions.*

Gauss in his computation of a table of the psi function succeeded in the explicit evaluation of $\Psi(p/q)$ and $\Psi(1-p/q)$ when p and q are integers. This paper shows how the theorem of Gauss is to be extended for the polygamma functions defined as the nth derivatives of $\Psi(x)$, $n=1, 2, \cdots$. (Received November 22, 1934.)

27. Professor Edward Kasner: *Differential equations admitting a conformal group.*

The author investigates differential equations of the first order $y' = f(x, y)$ which admit a continuous group of conformal transformations containing one or more parameters. An isothermal family of curves, for example, admits a three-parameter group. In the general case the author obtains a partial differential equation of the fourth order which must be satisfied by $f(x, y)$ if a one-parameter group is to exist. (Received November 19, 1934.)

28. Professor J. L. Walsh and Mr. J. H. Curtiss: *The Jacobi interpolation series on the lemniscate of convergence.*

By interpolation to a function $f(z)$ which is analytic at each of the m points $\alpha_1, \alpha_2, \cdots, \alpha_m$ of the complex z-plane the coefficients of a Jacobi series of the following type may be determined: $\sum_{h=0}^{\infty}\{[a_{hm}+\sum_{k=1}^{m-1}(a_{hm+k}\prod_{p=1}^{k}[z-\alpha_p])] \cdot (\omega(z))^h\}$, where $\omega(z)=\prod_{q=1}^{m}(z-\alpha_q)$. The present paper develops a method whereby the known results on the Taylor series can be applied to the study of this series on its lemniscate of convergence. The process depends basically on demonstrating that the function $f(z)$ can be uniquely expressed in the form $\phi_1(z)+\sum_{k=1}^{m-1}[\phi_{k+1}(z)\prod_{h=1}^{k}(z-\alpha_h)]$, where the functions $\phi_k(z)$ have the property that if $A_1(z)=z$, $A_2(z), \cdots, A_m(z)$ are the roots of the equation in t, $\omega(t) - \omega(z) = 0$, then $\phi_k(A_p(z)) = \phi_k(A_q(z))$, for all p, q. The paper then considers the various problems arising in the study of the Jacobi series on its lemniscate of convergence which can be attacked by this method and establishes results explicitly concerning uniform convergence, convergence in the mean, degree of

convergence, Abelian and Tauberian theorems, and the relation between the coefficients and the singularities of the function $f(z)$ on the leminscate. (Received November 21, 1934.)

29. Mr. W. E. Sewell: *Degree of approximation to continuous functions.*

If $f(z)$ is analytic in the unit circle and satisfies a Hölder condition of order α, $0<\alpha\leqq 1$, on the circumference, then $|f(z)-P_n(z)|<(M \log n)/n^{\alpha}$, where $P_n(z)$ is the sum of the first n terms of the Taylor development of $f(z)$ about $z=0$. This result is extended to regions bounded by arbitrary analytic Jordan curves, and by consideration of derivatives to an arbitrary $\alpha>0$, where the Taylor development is replaced by Faber's polynomials. The main result can be expressed in the following theorem: Let $f(z)$ be analytic in the region bounded by the analytic Jordan curve C, continuous in the corresponding closed region, and further let $|f^{(p)}(z_1)-f^{(p)}(z_2)|\leqq M|z_1-z_2|^{\alpha}$, $0<\alpha\leqq 1$, z_1, z_2 on C, p an arbitrary positive integer; then $|f(z)-\sum_0^n a_\nu P_\nu(z)|\leqq (M \log n)/n^{p+\alpha}$, where $P_\nu(z)$ is the Faber polynomial of degree ν belonging to the region and $f^{(p)}(z)$ denotes the pth derivative of $f(z)$. (Received November 21, 1934.)

30. Professor A. H. Copeland: *A mixture theorem concerning non-conservative mechanical systems.*

With the aid of the ergodic theorem, Eberhard Hopf obtains a mixture theorem for conservative mechanical systems. We shall investigate systems which are acted upon by friction in such a manner that they eventually come to rest. Let us consider a large number of experiments upon such a system. It is assumed that the initial conditions for this sequence of experiments are distributed in accordance with a given probability function. The distribution function for the final positions of the system is then uniquely determined. It is shown that in general, as the friction approaches zero, this latter distribution function approaches a constant times the Lebesgue measure function. That is, equal likelihood is approached as the limiting case. (Received November 20, 1934.)

31. Dr. Deane Montgomery (National Research Fellow): *Non-separable metric spaces.*

Several properties of metric spaces and functions defined on them which have so far been proved only under the hypothesis of separability are here demonstrated without this restriction. It is shown for example that if a set E is analytic locally then it is analytic throughout the entire space. (Received November 23, 1934.)

32. Professor C. C. Craig: *On the frequency function of xy.*

The object of this paper is to study the distribution of the product xy under the assumptions that x and y are independent and normally distributed. The moment generating function and the semi-invariants of Thiele are readily obtained. It is seen, in particular, that the skewness is never great but that the excess is often unusually high. It is noted that the form of the distribution de-

pends essentially upon two parameters, which are the ratios of the mean to the standard deviation of x and y respectively. The frequency function itself is obtained as an expansion in products of two Bessel functions. The series is shown to be convergent for all values of $z=xy$ except zero, at which the first term possesses a logarithmic discontinuity. (Received November 21, 1934.)

33. Mr. G. W. Petrie: *Probability of non-independent events.*

Luders has contributed a formula to the problem of the probability of non-independent events (Biometrika, May, 1934, pp. 108–128, *Die Statistik der seltenen Ereignisse*). He assumes that all events in which one cause produces exactly one event follow a Poisson law whose mean is h_1; all pairs of events, in which the two events are occasioned by one cause, follow a Poisson law whose mean is h_2; and, in general, all groups of n events which occur due to one cause follow a Poisson law whose mean is h_n. It is the purpose of this paper to describe an urn scheme from which Luders' formula is derived as a special case and to examine the conditions under which his formula holds as well as other conditions which seem equally likely in the problem of the probability of non-independent events. The following theorem is established: *If P_r is the probability of drawing r red balls in kn drawings without replacements from an urn containing n balls, pn of which are red and $(1-p)n$ of which are black, and p and n are allowed to vary so that their product is the constant m; then the limit of P_r as $p\to 0$ is given by the binomial law, $P_r={}_mC_rk^r(1-k)^{m-r}$.* (Received November 21, 1934.)

34. Dr. C. H. Fischer: *A sequence exhibiting correlation due to common elements.*

One random drawing of s_1 balls is made from an urn which is maintained so that the probability of drawing a white ball is constant, and a sequence of succeeding drawings of s_i, $(i=2, 3, \cdots, n)$, are made so that each s_i consists of t_i balls drawn at random from the preceding drawing plus s_i-t_i balls from the urn. The correlation between the number of white balls in any two drawings is explicitly determined by a priori methods and is shown to be independent of the constant probability of drawing a white ball from the urn. The linear equations of regression are found. Comparatively simple expressions for the multiple and partial correlation coefficients of any sets of successive drawings in the sequence are obtained. (Received November 22, 1934.)

35. Dr. J. L. Doob: *The theory of statistical estimation.*

If $f(x, p)$ is the density of distribution of the values of a chance variable $\mathbf{x}$ which depends on the parameter p, an important practical problem is that of determining p (supposed unknown) from a sample of values of $\mathbf{x}$. The amount of information in a sample pertinent to estimating p has been given an analytic expression by R. A. Fisher. There are many theorems needed to justify this expression. Thus if $\phi(x)$ is a single-valued function, it must be shown that the amount of information in a sample of values of $\mathbf{x}$ is no greater than the amount of information obtainable from a sample of values of $\phi(\mathbf{x})$. The purpose of this paper is to present proofs for such theorems. (Received November 22, 1934.)

36. Professor R. V. Churchill: *A comparison of the temperatures in a solid and its scaled model.*

If a solid is constructed of the same material and to scale with a body of any shape and subjected to initial and surface temperature conditions of the same character, properties can be derived to compare or contrast the temperatures at corresponding points in these solids. Seven properties are given in this paper, covering the cases of variable and steady temperatures of solids with and without radiation or convection at any part of the boundary, and with any fixed temperature distribution on the rest of the surface. The law used for the transfer of heat by radiation and convection is a very general one. The thermal coefficients are functions of the temperature. These properties are derived by substitutions of new variables into the heat equation and the boundary conditions. Some of the properties give important information about the effect of the size of a solid upon its rate of cooling or heating. (Received November 20, 1934.)

37. Mr. W. H. Erskine: *The Born-Infeld theory of electromagnetism.*

It is shown that the essential novelty, from the mathematical view point, of the Born-Infeld theory (Born and Infeld, Proceedings of the Royal Society, (A), vol. 144 (1934)) consists in a modification of the constitutive equations $\boldsymbol{B}=\boldsymbol{H}$, $\boldsymbol{E}=\boldsymbol{D}$, of Maxwell into a form where $\boldsymbol{B}$ and $\boldsymbol{E}$ are linear functions (still special) of the vectors $\boldsymbol{D}$ and $\boldsymbol{H}$ (the differential equations of Maxwell being unchanged). Various generalizations of the Lagrangean function proposed by Born and Infeld are considered and the resulting generalized constitutive equations and their implications are discussed. (Received November 22, 1934.)

38. Professor L. W. Cohen: *Transformations on spaces of infinitely many dimensions.*

The condition $\sum_i[\sum_k|a_{ik}|^{p/(p-1)}]^{p-1}<\infty$, $(p>1)$, determines a completely continuous transformation on H_p to H_p where H_p is the set of (x_i) for which $\sum_i|x_i|^p<\infty$. The condition $\sum_{i,k}|a_{ik}|^{r/(r-1)}<\infty$ where $p>1$, $q>1$, $r=\max(p, q/(q-1))$ determines a completely continuous transformation on H_p to H_q. The conditions $|a_{ik}|\leqq\sigma_i;\sum_i\sigma_i<\infty$ determine a completely continuous transformation on H_1 to H_1 and the determinant of $\delta_{ik}+a_{ik}$ is absolutely convergent. Properties of the minors are developed. The transposed determinant gives a theory of linear equations in $(x_i)\subset H_\infty$, the space where L.U.B. $|x_i|<\infty$. (Received November 19, 1934.)

39. Professor Dorothy McCoy: *Continuous transformations of finite homogeneous spaces.*

Stephens has studied the continuous transformations of certain finite spaces where a function determines a transformation of one space on another. This paper presents a study of such spaces under the additional requirement that both spaces be homogeneous. The possible combinations of a group of properties of the transformed spaces are studied exhaustively for several finite spaces. A large number of possibilities are found with some unusual relations due to

properties of the cardinal numbers of the spaces. (Received November 17, 1934.)

40. Dr. S. B. Myers (National Research Fellow): *Connections between differential geometry and topology.* I: *Simply connected surfaces.*

In this paper is presented a theory of new connections between differential geometry and topology. Given an arbitrary point A on a complete analytic Riemannian surface S, a point M on a geodesic ray g issuing from A is said to be a "minimum point with respect to A" on g if M is the last point on g such that AM furnishes an absolute minimum to the length of arcs joining A to M. The locus of such points with respect to A is proved to be a linear graph m which, if S is simply connected, is a tree when S is closed and a set of infinite trees when S is open. In a later paper, it will be proved that in the general case of a closed multiply connected S, m is a linear graph whose cyclomatic number is equal to the connectivity number mod 2 of S. The surface S is thus reduced to a single 2-cell a with m as its singular boundary; a is simply covered (except at A) by the geodesic rays through A cut off at their intersections with m. By means of m, and with the aid of the analyticity of S, relations are obtained between the coefficients of the Riemannian metric in the neighborhood of A and the topology of S. These methods present possibilities of generalization to n dimensions, to the geometry of paths in the large, and to convex metric spaces. (Received November 22, 1934.)

41. Professor C. N. Reynolds: *Properties of those spherical maps which may be colored in four colors.*

The four-color problem and the problem of so truncating, with triangular regions, some of the vertices of a cubic map as to make the number of regions bounding each region divisible by three are known to be equivalent. A map-coloring is accordingly defined as an infinite set of polyhedra equivalent with respect to the truncation of vertices and the inverse operation. In map-coloring in which $a_{n,r}$ regions have originally n sides and have $n+r$ sides after the truncation of vertices, we find that for suitably chosen rational functions $R_p(n, r)$, $\sum_{n,r} R_p(n, r)a_{n,r}$, $p = \cdots, -2, -1, 0, 1, 2, \cdots$, are relative invariants of order p under a certain class of truncations. These may be so combined as to form absolute invariants which in turn are used to develop properties of those spherical maps which may be colored in four colors. (Received November 22, 1934.)

42. Professor G. T. Whyburn: *Concerning the connectivity of limiting sets.*

In a compact metric space let the sequence of closed sets $[A_n]$ converge to the limiting set A. If for any $\epsilon > 0$ there exists a δ and an N such that for any $n > N$ it is true that any r-dimensional δ-cycle in A_n is ϵ-homologous to 0 in A_n, it is shown that A is an r-continuum in the sense that every r-dimensional Vietoris cycle in A is ~ 0 in A. Similar results are obtained when k ϵ-independent δ-cycles are allowed in A_n. A certain localization of the condition yields

the result that A is locally r-connected. In case each A_n is an irreducible membrane in the sense of Alexandroff, then with certain auxiliary convergence conditions it follows that A also is an irreducible membrane. The theorems obtained are applied to yield conditions under which A will be a simple arc or a 2-cell. (Received November 21, 1934.)

43. Dr. W. T. Reid: *A theorem on plane continua.*

In this paper the following theorem is proved: If M is a plane continuum and K is a proper sub-continuum of M, then at least one component of $M-K$ has a limit point in K. (Received November 22, 1934.)

44. Dr. W. T. Reid: *An integro-differential boundary problem.*

Let $I[\eta]=2H[\eta(a),\ \eta(b)]+\int_a^b 2\omega[x,\ \eta,\ \eta']dx+\int_a^b\int_a^b \eta_i(x)M_{ij}(x;\ t)\eta_j(t)dxdt$, where $\eta\equiv\{\eta_i(x)\}$ $(i=1,\cdots,n;\ a\leqq x\leqq b)$, H and ω are quadratic forms in $\eta_i(a)$, $\eta_i(b)$ and η_i, η_i', respectively, and $M_{ij}(x;\ t)=M_{ji}(t;\ x)$. This paper treats an integro-differential boundary problem associated with the problem of minimizing $I[\eta]$ in a class of arcs η which satisfy certain linear differential equations of the first order and certain linear end conditions, and which give a fixed value to the expression $2K[\eta]=2G[\eta(a),\ \eta(b)]+\int_a^b \eta_i(x)K_{ij}(x)\eta_j(x)dx+\int_a^b\int_a^b \eta_i(x)N_{ij}(x;\ t)\eta_j(t)dxdt$. It is supposed that G is a quadratic form in $\eta_i(a)$, $\eta_i(b)$, while $K_{ij}(x)=K_{ji}(x)$, $N_{ij}(x;\ t)=N_{ji}(t;\ x)$. The existence of infinitely many characteristic numbers for the boundary problem is established when the coefficients of K satisfy rather general conditions. There are also proved some expansion theorems in terms of the characteristic solutions. The method of proof is similar to that used previously by the author (American Journal of Mathematics, vol. 54 (1932), pp. 769–790) for the differential system that arises when $M_{ij}(x;\ t)\equiv 0\equiv N_{ij}(x;\ t)$, use now being made of the notion of a Green's matrix for an integro-differential boundary problem. (Received November 20, 1934.)

45. Dr. H. L. Krall: *Analytic continuation of solutions of elliptic differential equations.*

Let $U(x, y)$ satisfy the equation $U_{xx}+U_{yy}+a(x, y)U_x+b(x, y)U_y+c(x, y)U=f(x, y)$ in a region whose boundary contains an analytic arc σ, on which $U(x, y)$ satisfies the boundary condition $\partial U/\partial n=A(U, s)$, where $A(U, s)$ is an analytic function of U and the arc length s. The coefficients a, b, c, f are assumed to be analytic in the closed region. By means of a theorem of G. Giraud, it is shown that if $U(x, y)$ has continuous partial derivatives of the first eight orders in the closed region, it can be continued analytically over σ. (Received November 23, 1934.)

46. Mr. A. E. Heins: *A note on the equation of heat conduction.*

It is shown that the Laplace transform can be used to solve operationally certain linear partial differential equations with constant coefficients. Thus if $u(x_1, \cdots, x_k)$ has a transform $g(p_1, \cdots, p_k)$ then $\partial u/\partial x_i$ has a transform $p_i g(p_1, \cdots, p_k)$. With this in mind, the author proceeds to solve the equation $\Delta^2 u=(1/k)\partial u/\partial t$. The transform of this equation becomes $(p_1^2+p_2^2+p_3^2)g=(1/k)\partial g/\partial t$. This can be integrated by ordinary methods, and we can find

$g(p_1, p_2, p_3, t)$ as the transform of $u(x_1, x_2, x_3, t)$. Then by imposing initial conditions the unknown element is removed. For the case of heat flow in an infinite medium the transform of $g(p_1, p_2, p_3, t)$ is evaluated and the Faltung theorem is applied. Thus one arrives at the standard solution found in Goursat's *Cours d'Analyse*, vol. III, p. 107. (Received November 22, 1934.)

47. Mr. R. P. Bailey: *Coincidence of the formulas of mechanical quadratures of Gauss's type and of Tchebycheff's type.*

The formula of mechanical quadratures $\int_a^b f(x)p(x)dx = \sum_{i=1}^{n} H_i f(x_i) + R_n(f)$, $[p(x) \geqq 0 \text{ in } (a,b)]$, based on the Lagrange interpolation formula is said to be of Gauss's type if $R_n[G_{2n-1}(x)] = 0$, where $G_{2n-1}(x)$ is an arbitrary polynomial of degree $\leqq 2n-1$; it is of Tchebycheff's type, if the Cotes numbers H_i are all equal. In this note we show that these two conditions can hold at the same time only if (a, b) is finite and $p(x) = [(x-a)(b-x)]^{-1/2}$. (Received November 22, 1934.)

48. Dr. M. S. Robertson (National Research Fellow): *The coefficients of the Laurent series of a univalent function.*

It is well known that if $f(z) = \sum_1^\infty a_n z^n$, a_n real, $a_1 \neq 0$, is regular and univalent in the unit circle, then $|a_n| \leqq n|a_1|$. In this paper the author considers functions analytic in the annulus $R < |z| < 1$. If a_n, $(n = \pm 1, \pm 2, \cdots)$, are the coefficients of the Laurent series for $f(z)$ in this annulus, and if the a_n are real numbers, then it is shown that $|a_n - a_{-n}| \leqq n|a_1 - a_{-1}|$. (Received November 17, 1934.)

49. Dr. M. S. Robertson (National Research Fellow): *The radius of univalency of analytic functions.*

In this paper the author derives a formula for the radius of univalency of an analytic function $f(z)$ regular about the origin as the upper limit of certain determinants of increasing order n whose elements are made up from the coefficients of the corresponding power series for $f(z)$. From this formula the radii of univalency of certain analytic functions associated with a given analytic function of known radius of univalency R are determined in terms of R. (Received November 17, 1934.)

50. Dr. M. S. Robertson (National Research Fellow): *Functions regular and univalent in a sector.*

Functions $f(z)$ regular in a sector with vertex at the origin in the z-plane are studied with regard to the property of being univalent in this sector. An upper bound is obtained for the maximum aperture of the sector in which $f(z)$ can be univalent. For functions having a pole of order n at infinity the maximum aperture is found to be $2\pi/n$. The rates of growth of the function and its derivative along a radial line from the origin within the sector are obtained. (Received November 17, 1934.)

51. Dr. Rufus Oldenburger: *Equivalence of multilinear forms singular on one index.*

Necessary and sufficient conditions are obtained for the equivalence of multilinear forms singular on one index, and canonical forms are written down. (Received November 22, 1934.)

52. Dr. J. S. Frame: *On the irreducible representations of hyperorthogonal groups.* Preliminary communication.

A certain infinite family of simple groups of finite order may be represented by unitary matrices of degree m with coefficients from a finite field $GF(q^2)$, of q^2 elements. Here q is the power p^s of a prime p, and conjugates are defined by $\bar{x}=x^q$. Dickson denotes these groups by the symbol $HO(m, p^{2s})$. The vectors a, with components a_i from the $GF(q^2)$ for which $\sum_{i=1}^{m}\bar{a}_i a_i=0$, are permuted among themselves, and $\sum_{i=1}^{m}\bar{a}_i b_i$ is a bilinear invariant. In this paper we find for the case $m=3$ a set of $(q+1)q/(2d)$ monomial representations of these groups, of degree q^3+1 with complex coefficients, where d is the h.c.f. of m and $q+1$. Of these, $(q+1)/d$ split up, each into two irreducible components, and the rest are irreducible. By means of the relations between characteristics, we then determine the degrees of all the $q(q+1)/d+d+1$ irreducible representations of these groups, giving as well the classification of the elements into sets of conjugates. (Received November 21, 1934.)

53. Dr. Olga Taussky: *Fields as class fields of different reference fields.*

As is well known (Ph. Furtwängler was the first to prove it) the complete Hilbert class field of an algebraic number field is uniquely determined. The present paper discusses the converse fact that a given algebraic number field can be the complete Hilbert class field with respect to more than one of its subfields. (Received November 24, 1934.)

54. Professor Marston Morse: *Uniform instability and dynamical discontinua.*

This paper is concerned with a Riemannian manifold S, which is the image of a surface with a finite number of geodesically-convex boundary curves, and with a linear connectivity which exceeds 1. Each geodesic g is assumed to be of unstable type. This instability is defined in terms of the equation of normal variation from g. We assume further that this instability holds uniformly for all geodesics on S. Let M be the phase space defined by geodesic motion on S and let K be the point set on M swept out by a set H of complete trajectories on M. The set H is said to form a dynamical discontinuum if K contains no connected subsets other than points or segments of a single trajectory. We show that the set of all complete trajectories on M forms a dynamical discontinuum. (Received November 14, 1934.)

55. Professor G. C. Evans: *Condition that a harmonic function be a potential of positive mass.*

Let $u(M)$ be harmonic in a domain Σ (with bounded boundary s), not identically zero, and if Σ is an exterior domain, vanishing continuously at infinity. Let Σ' be a regular domain contained with its boundary s' in Σ and let $V'(M)$

be the function constituted by the solutions of the Dirichlet problems (interior or exterior, as the case may be) for each of the domains comprising $C(\Sigma'+s')$, with boundary values $u(M)$ on s'. A necessary and sufficient condition that $u(M)$ be given for all M in Σ as a potential of some distribution of positive mass is that for each Σ', $V'(M) \leqq u(M)$ for all M in $\Sigma-\Sigma'$. The mass may be placed entirely on s. (Received November 13, 1934.)

56. Professor C. N. Moore: *Convergence factors for series summable by Nörlund means of a certain type.*

We are given a set of constants $c_0 \neq 0$, c_1, c_2, $\cdots$, such that $\Sigma_0^n c_i = C_n$ becomes infinite with n. We further assume that the series $\Sigma C_n z^n$ converges in some circle of positive radius having the origin as center and that the series $\Sigma a_n z^n = (1/(\Sigma C_n z^n))$ is such that $\Sigma n|a_n| < \infty$. We say that a series Σu_n is summable (N^*) to σ if $\sigma_n = (S_n/C_n)$, where $S_n = c_n s_0 + \cdots + c_0 s_n$, $s_n = u_0 + \cdots + u_n$, $\to \sigma$ as $n \to \infty$. The theorems proved in the present paper are as follows. We have a set of functions $f_n(\alpha)(n=0, 1, 2, \cdots)$ defined over a set of points $E(\alpha)$ having a limit point α_0, not of the set. We require these functions to satisfy the condition (A): $|C_n f_{n+i}(\alpha)| < K(\alpha)$, where $K(\alpha)$ is bounded on $E(\alpha)$ for all n and i. We then define $Lf_n(\alpha) = a_0 f_n(\alpha) + a_1 f_{n+1}(\alpha) + \cdots$. If the $f_n(\alpha)$ satisfy the further condition (B): $\Sigma|C_n Lf_n(\alpha)| < K_1(\alpha)$, where $K_1(\alpha)$ is bounded on $E(\alpha)$, the series $\Sigma u_n f_n(\alpha)$ will converge on $E(\alpha)$. If further $\lim_{\alpha \to \alpha_0} f_n(\alpha) = 1$ for all n and $K(\alpha)$ and $K_1(\alpha)$ are each less in absolute value than a positive constant K over $E'(\alpha)$ including all points of $E(\alpha)$ in a certain neighborhood of α_0, then $\lim_{\alpha \to \alpha_0} \Sigma u_n f_n(\alpha) = \sigma$. (Received November 21, 1934.)

57. Professor Hans Rademacher: *A new estimate of the number of prime numbers in a real quadratic field.*

By a detailed investigation of zero-free regions in the critical strip of Hecke's $\zeta(s, \lambda)$-functions, and by the application of a certain Fourier series, it is possible to obtain an estimate for the number $\Omega_{\mathfrak{n}}(Y, Y')$ of prime numbers satisfying the conditions $0 < \omega \leqq Y$, $0 < \omega' \leqq Y'$, $\omega \equiv \rho \pmod{\mathfrak{n}}$. The error term is of the form $O(x \exp(-c(\log x)^{1/2}))$. In the course of the proof we can state at the same time improvements of some previous results obtained by Hecke with the use of his $\zeta(s, \lambda)$-functions and with the application of Weyl's method of diophantine approximation. (Received November 23, 1934.)

58. Professor D. V. Widder: *A classification of generating functions.*

In an earlier paper the author studied the operator $L_{k,t}[f(x)] = (-1)^k \cdot f^{(k)}(k/t)(k/t)^{k+1}/k!$. It was found to be useful in inverting the Laplace integral $\int_0^\infty e^{-xt} d\alpha(t)$. It is now found that this operator serves to give a more or less complete classification of the generating functions of such integrals. For example, it is shown that $f(x)$ has such a representation with $\alpha(t)$ of bounded variation in $(0, \infty)$ if and only if the integrals $\int_0^\infty |L_{k,t}[f(x)]|\, dt$ are uniformly bounded in k. Necessary and sufficient conditions are obtained for $f(x)$ to have such a representation with $\alpha(t)$ belonging to all important classes of functions, such as L^p, C^p, etc. In particular a necessary and sufficient condition for the

representation of a function by the most general non-absolutely convergent Laplace integral is obtained for the first time. By studying the case when $\alpha(t)$ is a step-function a necessary and sufficient condition for the representation of a function in a non-absolutely convergent Dirichlet series is also found. (Received November 21, 1934.)

59. Professor Richard Brauer: *On normal division algebras of prime degree.*

Given a field Z of degree p over a field of reference F. The question may be asked whether there exists a division algebra D of degree p over F for which Z is a splitting field, or, in other words, whether Z can be embedded in such division algebras D. Our answer is that this problem is completely equivalent to purely "commutative" problems. These are concerned with the possibility of embedding Z in fields of given Galois group. Our methods yield all the desired division algebras D, whenever any exist. (Received November 22, 1934.)

60. Professor C. M. Cramlet: *Integrability conditions of implicit differential equations.*

Systems of partial differential equations having r dependent and n independent variables: $F_j(x, z, \partial z/\partial x)=0$, are considered. It is shown that, if the complete system of eliminants of the second derivatives of these equations is found, and if this second set is treated similarly to form a third set, and so on, the chain will terminate in the integrable case and the integrability conditions become the conditions of algebraic consistency of the chain in the first derivatives. (Received December 1, 1934.)

61. Professor J. L. Walsh: *Interpolation and functions analytic interior to the unit circle.*

Let points $z_1, z_2, \cdots, (|z_k|<1)$, be given, and values $w_1, w_2, \cdots$. A necessary and sufficient condition that there exist a function $f(z)$ analytic and uniformly limited for $|z|<1$ and such that $f(z_k)=W_k$ is that the numbers M_n be uniformly limited, where M_n is a root of a certain algebraic equation and is the smallest number such that there exists a function $f_n(z)$ analytic and of modulus not greater than M_n for $|z|<1$, with $f_n(z_1)=w_1$, $f_n(z_2)=w_2, \cdots$, $f_n(z_n)=w_n$. The relation $\lim_{n\to\infty} M_n=\overline{\text{Bound}}_{|z|<1}|f(z)|$ is valid for a particular $f(z)$ if any $f(z)$ exists. Examples show that such a particular $f(z)$ if existent need not be unique. (Received December 1, 1934.)

62. Professor R. L. Jeffery: *Functions defined by sequences of integrals.*

If the sequence of functions $f_n(x)$ converges to the summable function $f(x)$, the sequence $\int_a^x f_n(x)dx$ may or may not tend to a limit. If it does tend to a limit $F(x)$, then the relation $F(x)=\int_a^x f(x)dx$ may or may not hold. The present note gives necessary and sufficient conditions for the existence of the limit function $F(x)$, and an analysis of the structure of $F(x)$ when this function is not equal to $\int_a^x f(x)dx$. It is also shown that certain general types of non-abso-

lutely convergent integrals can be expressed as the limit of a sequence of integrals. (Received December 3, 1934.)

63. Professor C. M. Cramlet: *On the problem of Pfaff.*

An existence theorem for the precise canonical form of a Pfaffian is given. The novel feature of the paper is in the fact that all conditions can be shown to be necessary and sufficient in advance of, instead of in virtue of, the actual reduction. (Received December 1, 1934.)

64. Professor C. M. Cramlet: *Differential invariant theory of alternating tensors.*

The alternating tensor $a_{r_1 \ldots r_q}$ of order q is studied. This tensor and another well known tensor (here called the derived tensor) involving the first derivatives only of the given tensor are shown to constitute a complete system of tensors. Unlike symmetric covariant tensors of order greater than one, the alternating tensor may not be used to establish covariant differentiation. The vanishing of the derived tensor is shown to be a necessary and sufficient condition that a coordinate system exist in which the components of the base tensor are constants. (Received December 1, 1934.)

65. Dr. C. C. Torrance: *Projections in abstract spaces.*

This paper generalizes the concept of a vector space. As ordinarily defined, an abstract vector can be multiplied by a real (or complex) number. This multiplication is here extended to the case where both factors are abstract elements. An inner product is introduced, but without the assumption that it is a complex number. A generalized form of Bessel's inequality is proved. (Received December 17, 1934.)

66. Professor Norbert Wiener: *The closure of Bessel functions*

A method from the theory of entire functions is given for the proof of the completeness of the set $J_n(\lambda_{mn}x)$, where $J_n(\lambda_{mn})=0$. The method is of wide applicability. (Received December 27, 1934.)

67. Professor John von Neumann: *On complete topological spaces.*

For a topological space in general the notion of completeness is based on the notion of metric. But for *linear* topological spaces the property of completeness can be defined independently, the property being that every topologically bounded set has a compact closure. And a set of the given linear space is called topologically bounded if it can be covered by a finite number of translations of any suitable neighborhood of the origin. (Received December 27, 1934.)

68. Professors Salomon Bochner and John von Neumann: *Almost periodic functions in groups.* II.

Von Neumann's notion of almost periodicity for functions in general groups is extended from numerical functions to "abstract" functions, the range-space being any topologically complete linear space. Furthermore, Bochner's

summation theorems for Fourier series are extended from Bohr's functions to numerical and abstract almost periodic functions in groups. A detailed discussion is given of almost periodic representations of arbitrary groups by unitary operators in Hilbert space. (Received December 27, 1934.)

69. Dr. Max Zorn: *Some remarks in the theory of algebraic functions.*

The Riemann-Roch theorem is shown to be a consequence of a theorem on moduli and of the theorem of the invariance of the differential class; for this invariance a new proof is given by elementary means. By introducing only topological conditions, an important part of Abel's theorem is proved, and the topological characterization of closed Riemann surfaces reduced to a euclidean problem. Finally, an abstract geometry on algebraic curves is defined by strictly combinatorial axioms. (Received December 3, 1934.)

70. Professor H. S. Wall: *Hypergroups.* Preliminary report.

Let $G(a_1, a_2, a_3, \cdots)$ be a countable set of distinct elements and $S(s_1, s_2, s_3, \cdots)$ the set of all distinct combinations of elements of G with repetitions. G is called a hypergroup if there is a rule of combination "$\circ$" such that: (I) If a_α, a_β are in G then there exist unique elements a_γ, a_δ in G such that $a_\alpha \circ a_\beta = (a_\gamma, a_\delta)$ (=an element s of S). (II) There is an element e in G such that, for every a in G, $e \circ a = a \circ e = (a, a)$. (III) For every a in G there is an "inverse" $\bar{a}$, a "left identity" a' and a "right identity" a'' such that $\bar{a} \circ a = (e, a')$, $a \circ \bar{a} = (e, a'')$. (IV) If $s_\mu = (a_\alpha, a_\beta, \cdots, a_\gamma)$, $s_\nu = (a_\rho, a_\sigma, \cdots, a_\tau)$ are elements of S then $s_\mu \circ s_\nu = (a_\alpha \circ a_\rho, a_\alpha \circ a_\sigma, \cdots, a_\alpha \circ a_\tau, \cdots, a_\gamma \circ a_\rho, a_\gamma \circ a_\sigma, \cdots, a_\gamma \circ a_\tau)$, which is an element of S. (V) If a_α, a_β, a_γ are in G, then $a_\alpha \circ (a_\beta \circ a_\gamma) = (a_\alpha \circ a_\beta) \circ a_\gamma$. The author proves (1) e is unique; (2) if $a_\alpha \circ a_\beta = a_\alpha \circ a_\gamma$ then $a_\beta = a_\gamma$ if and only if $a_\alpha' \circ a_\beta = a_\alpha' \circ a_\gamma$; (3) if $a_\alpha \circ a_\beta = a_\alpha \circ a_\gamma = a_\alpha \circ a_\delta$ then $a_\beta = a_\gamma = a_\delta$. If G is finite: (4) there exist matrices $A_1, A_2, A_3, \cdots$ such that if $a_\alpha \circ a_\beta = (a_\gamma, a_\delta)$ then $A_\alpha A_\beta = A_\gamma + A_\delta$; (5) the equations $a_\alpha \circ x = (a_\beta, y)$, $x \circ a_\alpha = (a_\beta, y)$ have solutions; (6) if the inverse is assumed unique then there is a finite group $B(b_1, c_1, d_1, e_1, b_2, c_2, d_2, e_2, \cdots)$ such that if $a_\alpha \circ a_\beta = (a_\gamma, a_\delta)$ then the distinct products of elements b_α, c_α, d_α, e_α times elememts b_β, c_β, d_β, e_β are b_γ, c_γ, d_γ, e_γ, b_δ, c_δ, d_δ, e_δ and no others. (Received December 1, 1934.)

71. Professor L. E. Dickson: *Cyclotomy and trinomial congruences.*

This paper presents a general theory of cyclotomy for $p = ef + 1 =$ prime, with details when e is a prime or the double of a prime. The number of solutions of $x^e + y^e \equiv \pm 1 \pmod{p}$ is given. The paper has been offered to the Transactions. (Received December 27, 1934.)

72. Professor Gabriel Szegö: *A problem concerning orthogonal polynomials.*

Let C and C' be two "weighted" Jordan curves with finite lengths, for which the corresponding sets of orthogonal polynomials are identical. The author shows that one of the curves (say C') contains the other (C), and is a level-curve

of the conformal representation of the region outside C onto the region outside of the unit circle, the weight-functions being the absolute values of the same analytic function regular outside C. He also shows that there are in all five cases in which the sets of orthogonal polynomials are the same for all the level-curves of a conformal representation. An enumeration of these cases is given. (Received December 27, 1934.)

73. Professor A. J. Maria: *A representation of the most general positive harmonic function in a bounded domain in three-dimensional euclidean space whose boundary points are conical.*

Let Σ be a bounded domain in three-dimensional euclidean space all of whose boundary points are conical; let $\mu(P)$ be any positive harmonic function in Σ of boundary t. The author shows that $\mu(P)=\int_t F(s;P)d\mu(e_s)$, where $F(s;P)$ is a fixed positive harmonic function of P in Σ and is continuous with respect to s. The function $\mu(e)$ is a completely additive not negative function of Borel measurable sets. Other results are obtained. (Received December 27, 1934.)

74. Professor A. J. Maria: *On the behavior of a harmonic function in the neighborhood of the boundary.*

Consider in three-dimensional euclidean space, for example, *any* domain of bounded boundary t of positive capacity. Let $u(P)=\int_t \bar{u}(s)dm(e_s;P)$, where $\bar{u}(s)$ is summable with respect to $m(e;P)$. If $\bar{u}(s)$ is continuous at Q, a regular point of t, then it is shown that $\lim_{P\to Q}u(P)=\bar{u}(Q)$. The author proves further that the class of functions $u(P)$ defined as above includes the class defined by Wiener in his Transactions article on the discontinuous boundary value problem. (Received December 27, 1934.)

75. Mr. Garrett Birkhoff: *On the lattice theory of ideals.* II.

Let R be any ring in which the chain condition holds. The ideals of R have unique reduced representations by means of irreducible components if and only if the lattice of the ideals of R is a distributive lattice. This new theorem can be proved purely abstractly, by lattice theory. (Received January 2, 1935.)

76. Dr. L. M. Blumenthal (National Research Fellow): *An algebraic proof for the quasi-congruence order of the R_n.*

A fundamental theorem in the metric characterization of the euclidean n-dimensional space R_n proves that if a semi-metric space R contains more than $n+3$ points and each subset of $n+2$ points of R is congruent with $n+2$ points of R_n, then the space R is congruent with a subset of R_n (Menger, Mathematische Annalen, vol. 100 (1928), p. 128). The proof given by Menger is geometrical and is based upon a previously developed theory of pseudo-euclidean $(n+3)$-tuples. To obtain the algebraic proof, the theorem is translated into one on symmetric determinants and the proof given is independent of metric geometry considerations. (Received December 1, 1934.)

77. Dr. L. M. Blumenthal (National Research Fellow): *A short proof of a theorem of K. Menger.*

Menger has proved that the determinant of a pseudo-euclidean $(n+3)$-tuple is not zero (Mathematische Annalen, vol. 100 (1928), p. 131). His proof is tedious, involving a limiting process and using the fact that the roots of an algebraic equation are continuous functions of the coefficients of the equation. We show that the theorem is an *immediate consequence* of a determinant expansion due to Jacobi (Baltzer, *Determinanten*, 5th edition, 1881, p. 63). An application of Jacobi's theorem gives also the sign of the determinant of a pseudo-euclidean $(n+3)$-tuple, a result not obtained by Menger's method. (Received December 1, 1934.)

78. Dr. A. B. Brown: *On certain analytic continuations and analytic homeomorphisms.*

We generalize to the case of n complex variables and one real variable a theorem of F. Severi regarding analytic continuation, over a region of finite diameter with connected boundary in the $(2n+1)$-space, of a function given analytic near the boundary. Severi proves the theorem only for the case that $n=1$ and the region is of simple type. The similar theorem for a function of $n>1$ complex variables and a region in the $2n$-space, an extension of a theorem of Hartogs, first stated by Osgood, is also completely established without restrictions as to simplicity of the region. As an application we prove theorems of similar scope for analytic homeomorphisms. (Received December 20, 1934.)

79. Dr. A. B. Brown: *On the locus of an analytic equation in the real plane.*

The author offers a proof of the following theorem, of which he has been unable to find a statement or proof in the literature. *Let $f(x, y)$ be analytic at the real point (x_0, y_0), with $f(x_0, y_0)=0$ and $f(x, y)$ irreducible at (x_0, y_0). Then the locus $f(x, y)=0$ near (x_0, y_0) in the real (x, y)-plane is of one of three types*, described roughly as follows: (1) *the point (x_0, y_0)*; (2) *a single smooth curve through (x_0, y_0)*; (3) *a cusp with vertex at (x_0, y_0)*. Details as to the structure in cases (2) and (3) are given. The locus for a reducible function $f(x, y)$ consists of a finite number of loci such as are described above, no two of which have any point in common except (x_0, y_0). (Received December 26, 1934.)

80. Professor J. W. Campbell: *Note on the clock problem in relativity.*

To illustrate the phenomenon of time distortion in relativity, the experience of a hypothetical observer who makes an extended excursion from the earth has frequently been cited. It has always, so far as the author knows, been claimed that such an observer would return to find he had lived a *shorter* time than those who had remained on the earth, and in commenting on the non-mutality of the experiences of the man who goes and those who stay Eddington has claimed that it is the man who is disturbed who lives the shorter time. In the present note it is shown, however, that disturbances do not

provide a fundamental criterion in the phenomenon, and in particular that an observer might make an extended excursion from the earth in such a way as to return and find that he had lived a *longer* time than those who had remained. The paper will appear in an early number of the Philosophical Magazine. (Received December 30, 1934.)

81. Dr. J. A. Clarkson and Dr. W. C. Randels: *Fourier series convergence criteria, as applied to continuous functions.*

Mazurkiewicz and Kaczmarz have shown the existence of continuous functions which fail to satisfy the Dini criterion for convergence at any point, and have pointed out that the set of such functions constitutes the complement of a set of the first category in the space of continuous functions. We show that the same is true of the Lebesgue criterion, and of several generalizations of the latter. (Received January 10, 1935.)

82. Dr. F. G. Dressel: *A generalization of harmonic functionals.*

The fundamental operation in the theory of harmonic functionals is that of forming the covariant derivative; the present paper replaces this operation by multiplication by a linear form, and arrives at a more general theory. Green's theorems for the new functionals are also included. (Received January 2, 1935.)

83. Mr. Aaron Fialkow: *Trajectories and lines of force.*

Let a particle start from rest in a positional field of force which fluctuates with the time. We define a curve, passing through the initial point in the direction of the particle's motion, which is uniquely and geometrically determined by the field of force and the starting time. If the field is stationary, this curve coincides with the single line of force through the initial point. Consider the ratio of the infinitesimal departures of the defined curve and the particle's trajectory from their common tangent. We call the limits of this ratio the *ratio set.* For stationary fields, Kasner proved that in general the ratio set is 3. For this same case, we have shown in a previous paper that the ratio set is always a closed interval and depends upon the order of contact between the line of force and its tangent. We now extend these results to fluctuating fields and show that the ratio set is always the sum of two closed intervals. In general the ratio set is 2. For analytic fields it is always one of the integers 2, 3, 4, $\cdots$. We also study the dependence of the ratio set on the starting time. (Received November 14, 1934.)

84. Mr. M. M. Flood: *Proper reduction of regular matric polynomials.*

The matric polynomial R is said to *right reduce* the polynomial D if the degree of their product DR is less than the degree of D. The reduction is *proper* if the leading coefficient of the product is regular. A linear basis is determined for the set of polynomials which right reduce a regular polynomial. In case D and R are both regular, it is shown that the degree of DR is greater than or

equal to the greatest degree invariant of the column vector space of D or the row vector space of R. In particular, for a given regular polynomial D, there always exist polynomials R which properly right reduce D so that the degree of DR is the same as the greatest degree invariant of the column vector space of D. (Received January 11, 1935.)

85. Mr. Marshall Hall: *Divisibility sequences of third order.*

Certain sequences u_n satisfying linear recurrences have the divisibility property $u_n \mid u_{mn}$. These include the sequences of Lucas, Lehmer, Poulet, and Pierce. This paper establishes some necessary conditions which a divisibility sequence must satisfy, and shows that there exists none of the third order based on a cubic field. (Received January 8, 1935.)

86. Mr. Marshall Hall: *Divisibility sequences of fourth order.*

A new type of divisibility sequence is exhibited, and its arithmetic properties are established. (Received January 8, 1935)

87. Mr. W. H. Ingram: *The dynamics of Gramme-ring generators.* Preliminary report.

It is shown that the Gramme-ring commutator machine is connected with a delta-connected slip-ring machine by a transformation of variable of Park-Blondel type to the moving coordinates of a previous note. The Kirchhoff equations of constraint of the second machine imply a null-function of the form $N \equiv \sum \lambda_k (dq_{k-1}/dt + i_k + dq_{k+1}/dt)$ in the activity function, where the λ's are Lagrangean multipliers, the q's true dynamical coordinates, and the i's the line currents. The commutator voltage for the first machine is given by $\partial N/\partial(d\xi/dt)$, where ξ is the appropriate quasi-coordinate. (Received December 27, 1934.)

88. Dr. Nathan Jacobson: *Rational methods in the theory of Lie algebras.*

The purpose of this note is to develop "rationally" that part of the theory of Lie algebras (infinitesimal groups) that centers around the Lie-Engel theorems. The method consists in determining the relation between the Lie algebra and an enveloping associative algebra generated by it. An example of this situation is the case of a Lie algebra L of linear transformations. The enveloping algebra A is then the minimal set of linear transformations which is closed under multiplication and contains L. The theorems of Lie follow from the result that if A is semi-simple then $L = S(+)L_1$ where S is abelian and L_1 is semi-simple. This result gives also an extension of a theorem of Cartan's on absolutely irreducible infinitesimal groups. The theorem of Engel is derived with the help of a trace argument due to van Kampen and to Weyl. (Received January 9, 1935.)

89. Dr. Nathan Jacobson: *Representation of Lie algebras.*

In this note the following lemma is proved: *If* H, E_+, *and* E_- *are three linear transformations such that* $[H, E_+] \equiv HE_+ - E_+H = 2E_+$, $[H, E_-] = -2E_-$, $[E_+, E_-] = H$, *then* E_+, E_- *are nilpotent and* H *has integral roots and simple ele-*

mentary divisors. Together with a theorem of Engel's this gives a generalization of Cartan's criteria for solvable and semi-simple Lie algebras (infinitesimal groups). The lemma is also used to give a simple algebraic proof of the theorem of complete reducibility of a simple Lie algebra of order 3 (the rotation group). Finally, the lemma is used to prove the diagonal form in any representation of the maximal abelian sub-algebra of a semi-simple Lie algebra. (Received January 9, 1935.)

90. Mr. F. B. Jones: *Some theorems concerning certain spaces of R. L. Moore.*

Let S be a space satisfying Axioms 0, 1, 2, 3 and 4 of R. L. Moore's *Foundations of Point Set Theory.* In this paper the author establishes the following three theorems: (I) If S is locally peripherally compact, then if P is a point of a region R, there exists a simple domain D containing P and lying in R. (II) If S is metric and the author's Axiom 5_1* (*Concerning R. L. Moore's Axiom* 5_1, this Bulletin, vol. 39 (1933), p. 671) holds true, S is completely separable. (III) If S is completely separable and the author's Axiom 5_1* holds true, S is homeomorphic with a subset of a completely separable space satisfying Moore's Axioms 0, 1, 2, 3 and 4, and in which it is true that if P is a point of a region R, there exists a simple domain D containing P and lying in R. (Received January 9, 1935.)

91. Dr. Solomon Kullback: *On samples from a multivariate normal population.*

Wilks has discussed generalized forms of the variance, correlation ratio, ratio of variances, and Hotelling's generalization of Student's distribution for samples from a multivariate normal population (Biometrika, vol. 24 (1932), pp. 471–494). The distributions of the first three are extremely complicated for samples from a population involving more than two variables. In this paper we present certain statistics which may be considered as generalizations of the variance, correlation ratio, ratio of variances, and Student's Z for samples from a multivariate normal population. The distributions of these statistics are found, using Wishart's generalized product moment distribution (Biometrika, vol. 20 A(1928), p. 32 ff.) and the theory of characteristic functions, and turn out to be well known forms which have been tabulated. (Received January 5, 1935.)

92. Dr. W. C. Randels: *On an approximate functional equation of Paley.*

Paley (Proceedings of the London Mathematical Society, (2), vol. 31)considers the function $f(z)=\sum_{n=1}^{\infty}b(n)\exp\{i\Delta(n)\}z^n$. He proves, under suitable restrictions on $b(x)$ and $\Delta(x)$, that $f(z)=\exp\{\pi i/4\}\cdot(2\pi)^{1/2}\sum_{\nu=1}^{\infty}[b\{n(\nu;\theta)\}/(\alpha'\{n(\nu;\theta)\}^{1/2}\exp\{i\Delta[n(\nu;\theta)]+i\theta n(\nu;\theta)-2\pi i\nu n(\nu;\theta)\}|z|^{n(\nu;\theta)}]+R(z)$, where $n(\nu;\theta)$ is a solution of the equation $\Delta(x)+\theta x-2\pi\nu x=0$. Under Paley's conditions $R(z)$ is continuous for $|z|=1$. In this paper Paley's restrictions are weakened and $R(z)$ is no longer continuous for $|z|=1$, but it is proved that the order of $R(z)$ is dominated by that of the sum as $|z|\to 1$. By means of this equation generalizations of results of Hardy and Littlewood, Hille, Ingham, and Wilton are obtained. (Received January 10, 1935.)

93. Mr. Moses Richardson: *The relative connectivities of symmetric products.* Preliminary report.

The previous work on symmetric products of P. A. Smith and the author is extended so as to yield explicit formulas for the so-called "relative connectivities" of the symmetric product of a complex. Application is made to Morse's theory of critical chords. (Received December 5, 1934.)

94. Dr. Abraham Sinkov: *A set of defining relations for the simple group of order* 1092.

This paper is essentially a continuation of a previous paper by H. R. Brahana entitled *Certain perfect groups generated by two operators of orders two and three*, the major portion of which is devoted to the relations $S^3=T^2=(ST)^7=1$. The groups defined by these relations are all perfect; consequently the order of the commutator of S and T is included as an additional defining relation. The present paper proves that the relations $S^3=T^2=(ST)^7=(S^{-1}T^{-1}ST)^7=1$ completely define the simple group of order 1092. It is then shown that this same group may be generated by 2 operators of order 6 whose product is of order 6, or by 2 operators of order 7 whose product is of order 7. Finally, the results of this paper, together with those obtained by Brahana, are used to show that the non-alternating simple group of order 20,160 can not be generated by 2 operators of orders 2 and 3. (Received January 4, 1935.)

95. Dr. Richard Trott: *Matrices permutable with the rational canonical form of a matrix.*

Let $A-\lambda E$ be a matrix with elements in a field K. Let the invariant factors of $A-\lambda E$ be $E_i(\lambda)$, and designate by $(p_i(\lambda))^{\eta_{ij}}$, $\eta_{ij}\geqq 1$, $(i=1, 2, \cdots, t)$, $(j=1, 2, \cdots, k_i)$, the powers of the distinct irreducible factors of $E_i(\lambda)$. We call $(p_i(\lambda))^{\eta_{ij}}$ the elementary divisors of $A-\lambda E$ with respect to K. Take as the diagonal block matrix $M=[M_1, M_2, \cdots, M_t]$, where $M_i=[N_{i1}, \cdots, N_{ik_i}]$ and $N_{ij}=(p_i, e, 0, \cdots, 0)$, where p_i is any matrix with elements in K, whose characteristic equation is $p_i(\lambda)=0$. (Unfamiliar notation used in this abstract is explained in the paper by Trott, American Journal of Mathematics, vol. 56, p. 360.) We show that if $PM=MP$, P is a diagonal block matrix $[P_1, P_2, \cdots, P_t]$, where P_i is of the same order as M_i and if $P_i=(P_{rs})$, $(r, s=1, 2, \cdots, k_i)$, P_{rs} is of the type $(f_0(p_i), \cdots, f_{s-1}(p_i))$, where $f_j(p_i)$ is a polynomial in the matrix p_i. (See Cullis, *Matrices and Determinoids*, vol. 3, part 1, pp. 456–459.) We deduce from this that if $PM=M'P$, P is a matrix, where P_{rs} is of the type $Q\{f_0(p_i), \cdots, f_{s-1}(p_i)\}$ and Q is a fixed symmetric matrix. (Received January 10, 1935.)

96. Professor F. M. Weida: *On the laws of distribution of certain statistics for samples of n from Poisson's first law of probability.*

This paper gives the following new results. We have obtained an explicit expression for the distribution of differences and for the distribution of quotients of variables which are themselves distributed according to the Poisson first law of probability. In the case of random samples of n drawn from a

universe whose law of distribution is the Poisson first law of probability, we have obtained an explicit expression for the law of distribution of sample standard deviations, for the law of distribution of sample geometric means, and for the law of distribution of sample harmonic means. (Received December 1, 1934.)

97. Professor Louis Weisner: *Abstract theory of inversion of finite sums.*

A system, which the author calls a hierarchy, is defined abstractly by means of a set of axioms. For any hierarchy a generalized Möbius function is defined, with the aid of which a generalized Dedekind inversion formula is obtained. A particularly important example of a hierarchy is the class of subgroups of a finite group. If $\alpha(G)$ and $\beta(G)$ are two group-theoretic functions such that $\beta(G)=\sum\alpha(D)$, where D ranges over the subgroups of G, $\alpha(G)$ can be expressed, by the inversion formula, in terms of $\beta(G)$ and the generalized Möbius function. (Received November 27, 1934.)

98. Professor Louis Weisner: *Some properties of prime-power groups.*

In the preceding paper entitled *Abstract theory of inversion of finite sums*, the author showed that inversion formulas, analogous to Dedekind's inversion formula, exist in any hierarchy. In the present paper the generalized Möbius function, in terms of which the inversion formulas are expressed, is evaluated for the hierarchy consisting of the subgroups of a prime-power group. A few properties of prime-power groups are derived with the aid of the inversion formulas. (Received January 8, 1935.)

99. Mr. Albert Whiteman: *On the law of quadratic reciprocity.*

In this paper the author gives a short proof of the quadratic reciprocity law which depends upon the following modified form of the Gaussian criterion due to Lange: If p is any integer not divisible by the odd prime q, then p is a quadratic residue or non-residue of q according as the number of least positive odd remainders in the series $2p, 4p, 6p, \cdots, (q-1)p$, mod q, is even or odd. It is shown that the number of odd remainders greater than p in the series $2\lambda p \pmod q$ is $\equiv(p-1)(q-1)/4 \pmod 2$. Then it is shown that the number of odd remainders less than p is exactly the same as the total number of odd remainders in the series $2\mu q \pmod p$. The theorem then follows immediately. (Received January 11, 1935.)

100. Professor E. P. Lane: *A canonical power series expansion for a surface.*

This paper provides a geometric description of a projective coordinate system in three-dimensional space, referred to which an analytic surface can be represented in the neighborhood of an ordinary non-parabolic point by an expansion of the form $z=x^2+y^2+x^3+Ay^3+Bx^4+Cy^4+\cdots$, the unwritten terms being of at least the fifth order. This expansion is used in discussing the loci of various osculants and other geometric elements associated with a point

of a plane curve of section of a surface made by a variable plane through a fixed tangent of the surface. In particular, among other things, it is shown that the loci of the eight-point nodal cubic, the eight-point cubic of Sannia, and the osculating cubic are algebraic surfaces of orders five, eight, and twelve respectively, and that the locus of the Halphen point is a unicursal curve of order seventeen. (Received December 5, 1934.)

101. Professor E. P. Lane: *Plane sections through an asymptotic tangent of a surface.*

The plane section of a surface made by a plane, not the tangent plane, through an asymptotic tangent at an ordinary non-parabolic point of the surface, has an inflexion at the point. Bompiani has enriched the projective differential geometry of a plane curve, in the neighborhood of an inflexion point on it, by introducing various osculants associated with the inflexion. It is the purpose of this note to study the loci of these osculants and other points and lines associated by Bompiani with the inflexion point of a plane curve of section of a surface made by a variable plane through a fixed asymptotic tangent of the surface. (Received December 14, 1934.)

102. Professor W. E. Roth: *On the bilateral equation in matrices.*

The present paper takes up the study of the equation $G(X) \equiv A_1 X^r B_1 + A_2 X_r^{-1} B_2 + \cdots + A_r X B_r + F = 0$ where A_i and B_i $(i = 1, 2, \cdots, r)$ are respectively $m \times n$ and $n \times p$ matrices, where F is an $m \times p$ matrix (all known), and where X is an unknown $n \times n$ matrix. The solution of this equation is accomplished by its reduction to that of a system of unilateral equations of the type studied elsewhere (Transactions of this Society, vol. 32 (1930), 61–80). The set to which the characteristic values of X, satisfying $G(X) = 0$, belong is determined. This set in general is not that of the values of λ for which the rank of $G(\lambda)$ is less than n. (Received December 20, 1934.)