ON THE EXPANSION COEFFICIENTS OF THE FUNCTIONS $u/\text{sn}u$ AND u^2/sn^2u †

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1. *Introduction.* It is well known that the elliptic function $\text{sn}(u, k)$ may be expanded in the form

$$\text{sn}(u, k) = \sum_{n=0}^{\infty} \frac{u^n}{n!} S_n(k^2),$$

where

(1) $$S_0(k^2) = 0, \quad S_1(k^2) = 1, \quad S_n(k^2) = \sum_{r=0}^{[(n-1)/2]} s_r(n)k^{2r}, \quad (n \geqq 2).$$

Moreover, $s_r(2j)=0$ for all values of r and j, and $s_r(2j+1)$ is an integer >0 for $0 \leqq r \leqq j$. Similarly we have

(2) $$\frac{u}{\text{sn}(u, k)} = \sum_{n=0}^{\infty} \frac{u^n}{n!} G_n(k^2),$$

where

$$G_0(k^2) = 1, \qquad G_n(k^2) = \sum_{r=0}^{[n/2]} g_r(n)k^{2r}, \qquad (n \geqq 1),$$

with $g_r(2j+1)=0$ for all values of r and j, and $g_r(2j) \neq 0$ for $0 \leqq r \leqq j$. In particular

(3) $$g_0(2) = g_1(2) = 1/3.$$

Again,

(4) $$\frac{u^2}{\text{sn}^2(u, k)} = \frac{1}{2} k^2u^2 + \sum_{n=0}^{\infty} \frac{u^n}{n!} T_n(k^2),$$

where

$$T_0(k^2) = 1, \qquad T_n(k^2) = \sum_{r=0}^{[n/2]} t_r(n)k^{2r}, \qquad (n \geqq 1),$$

with $t_r(2j+1)=0$ for all r and j, and $t_r(2j) \neq 0$ for $0 \leqq r \leqq j$. In particular,

(5) $$t_0(2) = 2/3, \quad t_1(2) = -1/3, \quad t_0(4) = -t_1(4) = t_2(4) = 8/5.$$

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The values of $s_r(n)$, $g_r(n)$ and $t_r(n)$ for $r=0, 1, 2, 3$, were given without proof by Hermite.† A proof for $s_r(n)$, for $r=0, 1, 2, 3$, was given by Gruder‡ but no proof for $g_r(n)$ and $t_r(n)$ seems to have been published. In this paper we shall prove the following results:

(6) $g_0(n) = -i^n(2^n - 2)B_n;$

(7) $g_1(n) = i^n 2^{-3} n(2(2^n - 2)B_n - 1 - (-1)^n);$

(8) $t_0(n) = -i^n(n-1)2^n B_n;$

(9) $t_1(n) = i^n n(n-1)2^{n-2}B_n;$

$$(10)\qquad \begin{aligned} t_2(2) &= 0,\\ t_2(n) &= -i^n n(n-1)((2n-7)2^{n-6}B_n - 2^{n-8}(1+(-1)^n)), \end{aligned}$$

where $n\neq 2$. The B_n are the Bernoulli numbers defined by the equation

$$(11)\qquad \frac{1}{2}u \operatorname{ctn}\frac{1}{2}u = \sum_{n=0}^{\infty}\frac{u^n}{n!}i^n B_n.$$

The proof depends on a number of recursion formulas for the Bernoulli numbers and the method may be applied to calculate further values of $g_r(n)$ and $t_r(n)$.

2. *Recursion Formulas for the Bernoulli Numbers.* To simplify the writing of formulas we adopt the notation

$$(f+g)^n = \sum_{\substack{\alpha+\beta=n\\ \alpha,\beta\geqq 0}}\sum \frac{n!}{\alpha!\beta!}f_\alpha g_\beta,$$

$$(f(*)+g(*))^n = \sum_{\substack{\alpha+\beta=n\\ \alpha,\beta\geqq 0}}\sum \frac{n!}{\alpha!\beta!}f(\alpha)g(\beta),$$

and similarly for $(f+g+h)^n$ and $(f(*)+g(*)+h(*))^n$. Let

$$(12)\qquad \begin{aligned} &A_n = (n-1)B_n, \qquad J_n = 2^n B_n, \qquad K_n = (n-1)2^n B_n,\\ &M_n = n(n-1)2^n B_n, \qquad P_n = n(n-1)(2n-7)2^n B_n,\\ &R_n = (2^n - 2)B_n, \qquad U_n = (n-1)(2^n-2)B_n,\\ &V_n = \tfrac{1}{2}n(1+(-1)^n), \qquad W_n = n(n-1)2^{n-1}(1+(-1)^n). \end{aligned}$$

† Collected Works, vol. 3, pp. 236–237.

‡ Wiener Sitzungsberichte, vol. 126, IIa (1917).

The results which we shall require are as follows:

$$(13)\qquad (B+B)^n = \begin{cases} -(n-1)B_n, & (n\neq 2),\\ -(n-1)B_n+\tfrac{1}{2}, & (n=2);\end{cases}$$

$$(14)\qquad 4(B+B+B)^n = 2(n-1)(n-2)B_n + n(n-1)B_{n-2};$$

$$(15)\qquad (R+R)^n = -(n-1)2^nB_n = -K_n;$$

$$(16)\qquad 2(R+R+R)^n = (n-1)(n-2)R_n - n(n-1)R_{n-2};$$

$$(17)\qquad (J+J)^n = \begin{cases} -(n-1)J_n, & (n\neq 2),\\ -(n-1)J_n+2, & (n=2);\end{cases}$$

$$(18)\qquad (B+J)^n = \begin{cases} -(n-1)B_n, & (n\neq 2),\\ -(n-1)B_n+1, & (n=2);\end{cases}$$

$$(19)\qquad 6(A+A)^n = -(n-2)(n-3)A_n + n(n-1)A_{n-2};$$

$$(20)\qquad 6(K+K)^n = -(n-2)(n-3)K_n + 4n(n-1)K_{n-2};$$

$$(21)\qquad \begin{aligned} 6(A+K)^n &= -(n-2)(n-3)A_n\\ &\quad + n(n-1)(3\cdot 2^{n-3}+1)A_{n-2};\end{aligned}$$

$$(22)\qquad \begin{aligned} -6(R+R+U)^n &= 6(K+U)^n\\ &= -(n-2)(n-3)U_n + n(n-1)U_{n-2};\end{aligned}$$

$$(23)\qquad -(R+R+V)^n = (K+V)^n = n(n-1)R_{n-2};$$

$$(24)\qquad (K+W) = \begin{cases} 4n(n-1)K_{n-2}, & (n\neq 4),\\ 4n(n-1)K_{n-2}-192, & (n=4);\end{cases}$$

$$(25)\qquad \begin{aligned} 6(K+P)^n &+ 24(M+M)^n + 6(P+K)^n\\ &= -(n-2)(n-3)P_n + 4n(n-1)P_{n-2}\\ &\quad + 32n(n-1)M_{n-2} - 24n(n-1)K_{n-2}.\end{aligned}$$

These formulas are readily proved and in most cases we merely indicate the method of proof.

Proof of (13). The function $x=(u/2)\ \mathrm{ctn}\ (u/2)$ satisfies the differential equations

$$(26)\qquad 4u\frac{dx}{du} = -4x^2+4x-u^2,$$

$$(27)\qquad 2u^2\frac{d^2x}{du^2} - 4u\frac{dx}{du} = 4x^3-4x+u^2x.$$

It follows from (11) and (26) that

$$4\sum_{n=1}^{\infty}\frac{u^n}{(n-1)!}i^nB_n = -4\sum_{n=0}^{\infty}\frac{u^n}{n!}i^n(B+B)^n$$
$$+4\sum_{n=0}^{\infty}\frac{u^n}{n!}i^nB_n + 2\frac{u^2}{2!}i^2.$$

Hence

$$8B_2 = -4(B+B)^2 + 4B_2 + 2,$$
$$4nB_n = -4(B+B)^n + 4B_n, \qquad (n \neq 2),$$

which proves (13). The formula (14) follows in the same way from (11) and (27).

Proof of (15). It is known that

$$-u\csc u = \sum_{n=0}^{\infty}\frac{u^n}{n!}i^nR_n.$$

Since $u^2 \csc^2 u = u^2 + u^2 \operatorname{ctn}^2 u$, we have

$$\sum_{n=0}^{\infty}\frac{u^n}{n!}i^n(R+R)^n = -2\frac{u^2}{2!}i^2 + \sum_{n=0}^{\infty}\frac{u^n}{n!}i^n2^n(B+B)^n,$$

$$(R+R)^n = \begin{cases} 2^n(B+B)^n = -(n-1)2^nB_n, & (n \neq 2),\\ -2+2^2(B+B)^2 = -(2-1)2^2B_2, & (n=2).\end{cases}$$

Proof of (16). The function $y = u \csc u$ satisfies the equation

$$u^2\frac{d^2y}{du^2} - 2u\frac{dy}{du} = 2y^3 - 2y - u^2y.$$

Hence

$$-n(n-1)R_n + 2nR_n = -2(R+R+R)^n + 2R_n$$
$$- n(n-1)R_{n-2},$$
$$2(R+R+R)^n = (n-1)(n-2)R_n - n(n-1)R_{n-2}.$$

Proof of (17). In (13) replace B_n by $2^{-n}J_n$.

Proof of (18). We have

$$\frac{1}{2}u^2\operatorname{ctn}\frac{1}{2}u\operatorname{ctn}u = \sum_{n=0}^{\infty}\frac{u^n}{n!}i^n(B+J)^n,$$

and

$$\frac{1}{2}u^2 \operatorname{ctn}\frac{1}{2}u \operatorname{ctn} u = \frac{1}{4}u^2 \operatorname{ctn}\frac{1}{2}u - \frac{1}{4}u^2$$
$$= \frac{1}{2}\frac{u^2}{2!}i^2 + \sum_{n=0}^{\infty}\frac{u^n}{n!}i^n(B+B)^n.$$

Hence

$$(B+J)^n = \begin{cases} (B+B)^n = -(n-1)B_n, & (n \neq 2), \\ \frac{1}{2} + (B+B)^2 = -(n-1)B_n + 1, & (n = 2). \end{cases}$$

PROOF OF (19). The function

$$z = \frac{1}{4}u^2 \csc^2\frac{1}{2}u = \sum_{n=0}^{\infty}\frac{u^n}{n!}i^n 2^{-n}(R+R)^n = -\sum_{n=0}^{\infty}\frac{u^n}{n!}i^n A_n$$

satisfies the differential equation

$$u^2\frac{d^2z}{du^2} - 4u\frac{dz}{du} = 6z^2 - 6z - u^2 z.$$

Hence

$$-n(n-1)A_n + 4nA_n = 6(A+A)^n + 6n - n(n-1)A_{n-2}.$$

PROOF OF (20). In (19) replace A_n by $2^{-n}K_n$.

PROOF OF (21). We have

$$\frac{1}{4}u^4 \csc^2\frac{1}{2}u \csc^2 u = \sum_{n=0}^{\infty}\frac{u^n}{n!}i^n(A+K)^n,$$

and

$$\frac{1}{4}u^4 \csc^2\frac{1}{2}u \csc^2 u = \frac{1}{16}u^4 \csc^4\frac{1}{2}u + \frac{1}{4}u^4 \csc^2 u$$
$$= \sum_{n=0}^{\infty}\frac{u^n}{n!}i^n(A+A)^n - \frac{1}{4}\sum_{n=0}^{\infty}\frac{u^{n+2}}{n!}i^n 2^n A_n.$$

Hence

$$6(A+K)^n = 6(A+A)^n + 3\cdot 2^{n-3}n(n-1)A_{n-2}.$$

PROOF OF (22). The fact that $-(R+R+U)^n = (K+U)^n$ follows from (15). Moreover, $U_n = K_n - 2A_n$. Hence, from (20), (21), and (12),

$$\begin{aligned}6(K+U)^n &= 6(K+K)^n - 12(A+K)^n \\ &= -(n-2)(n-3)K_n + 4n(n-1)K_{n-2} \\ &\quad + 2(n-2)(n-3)A_n \\ &\quad - 2n(n-1)(3\cdot 2^{n-3}+1)A_{n-2} \\ &= -(n-2)(n-3)U_n + n(n-1)U_{n-2}.\end{aligned}$$

Proof of (23). As before, $-(R+R+V)^n=(K+V)^n$. Moreover, we have

$$u \sin u = -\sum_{n=0}^{\infty} \frac{u^n}{n!}\frac{1}{2} n(i^n+(-i)^n) = -\sum_{n=0}^{\infty}\frac{u^n}{n!} i^n V_n.$$

Hence

$$u^3 \csc^2 u \sin u = \sum_{n=0}^{\infty}\frac{u^n}{n!} i^n (K+V)^n$$

and

$$u^3 \csc^2 u \sin u = u^3 \csc u = -\sum_{n=0}^{\infty}\frac{u^{n+2}}{n!} i^n R_n.$$

Then we have $(K+V)^n = n(n-1)R_{n-2}$.

Proof of (24). We have

$$\begin{aligned}4u^2 \cos 2u &= -\sum_{n=0}^{\infty}\frac{u^n}{n!} n(n-1)2^{n-1}(i^n+(-i)^n) \\ &= -\sum_{n=0}^{\infty}\frac{u^n}{n!} i^n W_n.\end{aligned}$$

Then

$$4u^4 \csc^2 u \cos 2u = \sum_{n=0}^{\infty}\frac{u^n}{n!} i^n (K+W)^n,$$

and

$$\begin{aligned}4u^4 \csc^2 u \cos 2u &= 4u^4 \csc^2 u - 8u^4 \\ &= -192\frac{u^4}{4!} i^4 - 4\sum_{n=0}^{\infty}\frac{u^{n+2}}{n!} i^n K_n.\end{aligned}$$

Hence

$$(K + W)^n = \begin{cases} 4n(n-1)K_{n-2}, & (n \neq 4), \\ 4n(n-1)K_{n-2} - 192, & (n = 4). \end{cases}$$

PROOF OF (25). Since

$$K_\alpha P_\beta + 4M_\alpha M_\beta + P_\alpha K_\beta = (\alpha + \beta)(2\alpha + 2\beta - 7)K_\alpha K_\beta,$$

we have, from (20) and (12),

$$\begin{aligned} 6(K + P)^n &+ 24(M + M)^n + 6(P + K)^n \\ &= 6n(2n - 7)(K + K)^n \\ &= n(2n - 7)\{-(n-2)(n-3)K_n + 4n(n-1)K_{n-2}\} \\ &= -(n-2)(n-3)P_n + 4n(n-1)P_{n-2} \\ &\quad + 32n(n-1)M_{n-2} - 24n(n-1)K_{n-2}. \end{aligned}$$

3. *The Proof of* (6) *and* (7). Since $\operatorname{sn}(u, 0) = \sin u$, we have, from (2) and (12),

$$\sum_{n=0}^{\infty} \frac{u^n}{n!} g_0(n) = -\sum_{n=0}^{\infty} \frac{u^n}{n!} i^n R_n,$$

$$g_0(n) = -i^n R_n = -i^n(2^n - 2)B_n.$$

Next, the function $w = \operatorname{sn}(u, k)$ satisfies† the equations

$$\left(\frac{dw}{du}\right)^2 = (1 - w^2)(1 - k^2w^2), \quad \frac{d^2w}{du^2} = -(1 + k^2)w + 2k^2w^3.$$

It follows that $\xi = u/\operatorname{sn}(u, k) = u/w$ satisfies the equation

$$u^2 \frac{d^2\xi}{du^2} - 2u\frac{d\xi}{du} = 2\xi^3 - 2\xi - (1 + k^2)u^2\xi. \tag{28}$$

Hence, from (2),

$$\begin{aligned} n(n-1)G_n - 2nG_n &= 2(G + G + G)^n - 2G_n \\ &\quad - (1 + k^2)n(n-1)G_{n-2}, \end{aligned}$$

$$2(G + G + G)^n = (n-1)(n-2)G_n + (1 + k^2)n(n-1)G_{n-2},$$

$$\begin{aligned} 2\sum_{\substack{\lambda+\mu+\nu=r \\ \lambda,\mu,\nu \geqq 0}}\sum\sum &(g_\lambda(*) + g_\mu(*) + g_\nu(*))^n \\ &= (n-1)(n-2)g_r(n) + n(n-1)(g_r(n-2) + g_{r-1}(n-2)). \end{aligned}$$

† Gruder, loc. cit.

When $r=1$, the last equation becomes

$$(29)\quad \begin{aligned} 6(g_0(*) + g_0(*) + g_1(*))^n = (n-1)(n-2)g_1(n) \\ + n(n-1)(g_1(n-2) + g_0(n-2)). \end{aligned}$$

Now let

$$Z_n = i^n 2^{-3} n(2(2^n - 2)B_n - 1 - (-1)^n) = i^n 2^{-2}(U_n + R_n - V_n)$$

and consider $6(g_0(*)+g_0(*)+Z)^n$. By (16), (22), and (23), this equals

$$\begin{aligned} i^n 2^{-2}&\{6(R+R+U)^n + 6(R+R+R)^n - 6(R+R+V)^n\} \\ &= i^n 2^{-2}\{(n-2)(n-3)U_n - n(n-1)U_{n-2} \\ &\quad + 3(n-1)(n-2)R_n - 3n(n-1)R_{n-2} + 6n(n-1)R_{n-2}\} \\ &= i^n 2^{-2}\{(n-1)(n-2)(U_n + R_n - V_n) \\ &\quad - n(n-1)(U_{n-2} + R_{n-2} - V_{n-2}) + 4n(n-1)R_{n-2}\} \\ &= (n-1)(n-2)Z_n + n(n-1)Z_{n-2} + n(n-1)g_0(n-2). \end{aligned}$$

This proves that Z_n satisfies the recursion formula (29). Since, from (3), we have $g_1(2)=1/3=Z_2$, it follows that $g_1(n)=Z_n$ for all values of $n \geqq 2$. It is readily verified that $g_1(0)=Z_0=g_1(1)=Z_1=0$, and this completes the proof.

4. *The Proof of* (8), (9), (10). We have

$$\sum_{n=0}^{\infty} \frac{u^n}{n!} t_0(n) = \frac{u^2}{\mathrm{sn}^2(u, 0)} = u^2 \csc^2 u = -\sum_{n=0}^{\infty} \frac{u^n}{n!} i^n K_n,$$

$$t_0(n) = -i^n K_n = -i^n(n-1)2^n B_n.$$

Next, we have†

$$(-1)^r t_r(2j) = \sum_{\nu=0}^{r} \binom{j-\nu}{j-r} t_\nu(2j),$$

$$t_1(2j) = -\tfrac{1}{2} j t_0(2j) = i^{2j} 2^{-2} 2j(2j-1)2^{2j} B_{2j}.$$

Since $t_1(2j+1)=B_{2j+1}=0$, this proves (9). Again, the function $\eta = u^2/\mathrm{sn}^2(u, k) - k^2u^2/2$ satisfies the equation

$$u^2 \frac{d^2\eta}{du^2} - 4u \frac{d\eta}{du} = 6\eta^2 - 6\eta - 2(2-k^2)u^2\eta - k^2 \frac{1}{2} u^4.$$

† Gruder, loc. cit.

Hence, from (4),

$$n(n-1)T_n - 4nT_n = 6(T+T)^n - 6T_n - 2(2-k^2)n(n-1)T_{n-2}, \quad (n \neq 4),$$

$$6\sum_{\substack{\lambda+\mu=r \\ \lambda,\mu\geqq 0}}\sum (t_\lambda(*)+t_\mu(*))^n = (n-2)(n-3)t_r(n) + 4n(n-1)t_r(n-2) - 2n(n-1)t_{r-1}(n-2), \quad (n \neq 4).$$

When $r=2$, the last equation becomes

$$\begin{aligned} &6(t_0(*)+t_2(*))^n + 6(t_1(*)+t_1(*))^n + 6(t_2(*)+t_0(*))^n \\ (30) \qquad &= (n-2)(n-3)t_2(n) + 4n(n-1)t_2(n-2) \\ &\quad - 2n(n-1)t_1(n-2), \qquad (n \neq 4). \end{aligned}$$

Let

$$\begin{aligned} H_n &= -i^n n(n-1)((2n-7)2^{n-6}B_n - 2^{n-8}(1+(-1)^n)) \\ &= -i^n 2^{-6}P_n + i^n 2^{-7}W_n, \qquad (n \neq 2). \\ H_2 &= -i^2 2^{-6}P_2 - i^2 2^{-7}W_2 = 0\cdot \end{aligned}$$

and consider

$$6(t_0(*)+H)^n + 6(t_1(*)+t_1(*))^n + 6(H+t_0(*))^n \qquad (n \neq 4).$$

By (25), (24), and (12), this equals

$$\begin{aligned} &i^n 2^{-6}\{6(K+P)^n + 24(M+M)^n + 6(P+K)^n\} \\ &\quad - i^n 2^{-7}12\{(K+W)^n - n(n-1)K_{n-2}W_2\} \\ &= i^n 2^{-6}\{-(n-2)(n-3)P_n + 4n(n-1)P_{n-2} \\ &\quad + 32n(n-1)M_{n-2} - 24n(n-1)K_{n-2}\} \\ &\quad - i^n 2^{-7}12\{4n(n-1)K_{n-2} - 8n(n-1)K_{n-2}\} \\ &= i^n 2^{-6}\{-(n-2)(n-3)(P_n - 2^{-1}W_n) \\ &\quad + 4n(n-1)(P_{n-2} - 2^{-1}W_{n-2}) + 32n(n-1)M_{n-2}\} \\ &= (n-2)(n-3)H_n + 4n(n-1)H_{n-2} - 2n(n-1)t_1(n-2). \end{aligned}$$

This shows that H_n satisfies the recursion formula (30). Since $t_2(4) = 8/5 = H_4$, it follows that $t_2(n) = H_n$ for all $n \geqq 4$. Finally, we verify that $t_2(n) = H_n = 0$ for $n = 0, 1, 2, 3$. This completes the proof of (8), (9), and (10).

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