ON THE MINIMIZING PROPERTY OF THE HARMONIC FUNCTION

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1. *Introduction.* Let D be a bounded connected open set and D^* its boundary and let $\bar{D}=D+D^*$. It is well known † that if the function $u(x, y)$ be harmonic on D, then $u(x, y)$ minimizes the Dirichlet integral

$$I[f] = \int\int_D [f_x^2 + f_y^2]dx\,dy$$

in the class of all functions $f(x, y)$ possessing piecewise continuous partial derivatives f_x and f_y and coinciding with $u(x, y)$ on the boundary D^*. But in certain recent discussions of the problem of Plateau‡ essential use is made of a generalization of this theorem; it was necessary to know that the harmonic function $u(x, y)$ minimizes $I[f]$ in a larger class of functions than those with continuous derivatives. It has been suggested that a proof of this fact should be published; the present note carries out the suggestion. The method of proof is similar to that due to Lebesgue. As an application, a theorem is proved which is of some interest in the theory of curved surfaces.

The functions with which we shall be concerned are those which are, as we shall say, *absolutely continuous by sections* (abbreviated a.c.s.). A function $v(x, y)$, defined and continuous on a bounded open set D, will be said to be a.c.s. on D if it satisfies the following conditions:

(1 a) *for almost all values y_0 of y the function $v(x, y_0)$ is absolutely continuous on each interval of the line $y=y_0$ lying in D;*

(1 b) *for almost all values x_0 of x it is absolutely continuous on each interval of the line $x=x_0$ lying in D.*

† H. Lebesgue, Société Mathématique de France, Comptes Rendus, (1913), p. 48. Hurwitz-Courant, *Funktionentheorie*, 2d ed., p. 424.

‡ E. J. McShane, *Parametrizations of saddle surfaces*, etc., Transactions of this Society, vol. 35 (1933), pp. 716–733. T. Radó, *The problem of Plateau*, vol. II, No. 3, of the Ergebnisse der Mathematik und ihre Grenzgebiete, p. 99.

With this terminology we state our theorem on the minimizing property of the harmonic function.

THEOREM 1. *If* (a) *u is continuous on* $\bar{D}$ *and harmonic on* D, (b) *v is continuous on* $\bar{D}$ *and a.c.s. on* D, (c) $v(x, y)=u(x, y)$ *on* D^*, *then* $I[v]\geqq I[u]$, *the sign of equality holding only if* $I[u]=\infty$ *or* $v\equiv u$.

Before proceeding to the proof of the theorem, we first observe that if $f(x, y)$ be a.c.s. on D, its derivatives v_x and v_y are defined almost everywhere on D and are measurable where defined. Where they are undefined, we assign them the value 0. We shall denote by X the set of all values x_0 of x such that (a) the line $x=x_0$ has points in common with D, (b) on each interval of the line $x=x_0$ lying in D the function $v(x_0, y)$ is absolutely continuous in y. The set Y is defined analogously. For each x_0 of X the line $x=x_0$ has in common with D a finite or denumerable set of intervals; these we denote by $\delta_1(x_0)$, $\delta_2(x_0)$, $\cdots$. The intervals $\delta_i(y_0)$ are analogously defined. As a first step in the proof of Theorem 1 we establish the following lemma.

LEMMA. *If* $u(x, y)$ *is harmonic on an open set containing* $\bar{D}$, *and v is continuous on* $\bar{D}$ *and a.c.s. on* D, *and* $v(x, y)=u(x, y)$ *on* D^*, *then* $I[v]\geqq I[u]$.

If $I[v]=\infty$, the conclusion holds. Otherwise let us define

$$\phi(x, y)=v(x, y)-u(x, y).$$

This function is continuous on $\bar{D}$ and a.c.s. on D, and

$$I[v] = I[u] + I[\phi] + 2\iint_D (u_x\phi_x + u_y\phi_y)dx\,dy; \tag{1}$$

here all integrals are well defined and finite. We may write

$$\iint_D u_x\phi_x dx\,dy = \int_Y\left\{\int_{\Sigma\delta_i(y)} u_x\phi_x dx\right\}dy$$
$$= \int_Y\left\{\sum\int_{\delta_i(y)} u_x\phi_x dx\right\}dy.$$

In the last integral we integrate by parts, remembering that ϕ vanishes at each end of each $\delta_i(y)$. We thus obtain

$$\iint_D u_x \phi_x dx\,dy = -\int_Y \left\{ \sum \int_{\delta_i(y)} u_{xx}\phi\,dx \right\} dy$$
$$(2) \qquad = -\iint_D u_{xx}\phi\,dx\,dy;$$

the last reduction is allowable because u_{xx} and ϕ are both continuous and bounded on $\bar{D}$. In a like manner,

$$\iint_D u_y \phi_y dx\,dy = -\iint_D u_{yy}\phi\,dx\,dy.$$

Adding and remembering that u is harmonic, we find that the last integral of equation (1) vanishes. Since $I[\phi] \geqq 0$, the lemma is established.

Proceeding to the proof of the theorem, we subdivide D into the sets D_0, D', D'', on which the respective relations $v=u$, $v>u$, $v<u$ hold. Also, for every $\epsilon>0$, we define D_ϵ' and D_ϵ'' to be the subsets of D on which $v>u+\epsilon$ and $v<u-\epsilon$, respectively. The Dirichlet integrals over these sets will be distinguished by the corresponding affixes; for example, $I_\epsilon'(u)$ is the integral of $u_x^2+u_y^2$ over D_ϵ'.

We first observe that $I_0[v]=I_0[u]$; in fact, $u_x=v_x$ and $u_y=v_y$ for almost all points of D_0. For let E be the set on which $u-v=0$, the derivatives u_x and v_x are defined, and $u_x-v_x\neq 0$. This set is measurable; hence to prove that its measure is zero, we need only show that for almost all y_0 the part of E lying on the line $y=y_0$ has linear measure zero. But for every y_0 the points of E belonging to the line $y=y_0$ form an isolated set and so are enumerable; for at each such point we have $u-v=0$, while $\partial(u-v)/\partial x$ exists and is not zero. Therefore $m(E)=0$.

As ϵ tends to 0, the sets D_ϵ', D_ϵ'' tend to D', D'', respectively; hence $I_\epsilon'[u]\rightarrow I'[u]$ and $I_\epsilon''[u]\rightarrow I''[u]$. Thus if h be any number less than $I[u]$, we can choose ϵ small enough so that $I_\epsilon'[u]+I_\epsilon''[u]+I_0[u]>h$. The set D_ϵ' lies, with its boundary $D_\epsilon'^*$, in D; and on $D_\epsilon'^*$ we have $v-\epsilon=u$. Hence by our lemma $I_\epsilon'[v]=I_\epsilon'[v-\epsilon]\geqq I_\epsilon'[u]$, and similarly $I_\epsilon''[v]\geqq I_\epsilon''[u]$. Thus

$$I[v] \geqq I_0[v] + I_\epsilon'[v] + I_\epsilon''[v] \geqq I_0[u] + I_\epsilon'[u] + I_\epsilon''[u] > h.$$

This being true for every $h<I[u]$, it follows that $I[v]\geqq I[u]$, as was to be proved.

It remains to be shown that if $I[u]$ is finite and $I[v]=I[u]$, then $v\equiv u$. In this case we consider the function

$$I[u+\lambda\phi] = I[u] + \lambda^2 I[\phi] + 2\lambda \int\int_D (u_x\phi_x + u_y\phi_y)dx\,dy,$$

where $\phi=v-u$. On setting $\lambda=1$, we find that the last integral on the right has the value $-I[\phi]/2$. Setting $\lambda=1/2$, the equation becomes $I[u+\phi/2]=I[u]-I[\phi]/4$. Therefore, if $I[\phi]>0$, the a.c.s. function $u+\phi/2$ has a Dirichlet integral less than that of u, which we have already shown impossible; hence $I[\phi]=0$. This implies that $\phi_x=0$ almost everywhere in D. Thus for almost all y_0 of Y the equation $\phi_x(x, y_0)=0$ holds for almost all x in $\sum\delta_i(y_0)$; integrating and remembering that $\phi=0$ at the ends of each $\delta_i(y_0)$, we find $\phi(x, y)=0$ for almost all (x, y) in D. Since ϕ is continuous, $\phi=u-v$ vanishes identically on D, completing the proof of the theorem.

2. *Discussion of a.c.s. Functions.* As yet we have not shown that the class of a.c.s. functions includes the class of functions with piecewise continuous derivatives and finite Dirichlet integral. Suppose, then, that $v(x, y)$ and its derivatives v_x and v_y are continuous on D, and $I[v]$ is finite; we state that v is a.c.s. For the finiteness of $I[v]$ implies that $|v_x(x, y_0)|$ is summable over $\sum\delta_i(y_0)$ for almost all y_0. Since v_x is continuous in D, v is absolutely continuous over every closed interval contained in $\delta_i(y_0)$; and this, with the summability of $v_x(x, y_0)$, implies that v is absolutely continuous on $\delta_i(y_0)$. A like argument applies to $v(x_0, y)$ for almost all x_0. In particular, if $u(x, y)$ is harmonic on D and $I[u]$ is finite, then $u(x, y)$ is a.c.s.

If v is continuous on D and $I[v]$ is finite, and D can be subdivided into a finite number of subsets, each bounded by a finite number of rectifiable simple arcs such that v_x and v_y are continuous on the interior of each subset, then v is a.c.s. For, if we except a set of x_0 of measure zero, the line $x=x_0$ intersects the boundary curves in a finite number of points, so that each $\delta_i(x_0)$ is subdivided into a finite number of sub-intervals. Again except on a set of measure zero, the function $v(x_0, y)$ is absolutely continuous on each of these subintervals by the preceding paragraph, and being also continuous on $\delta_i(x_0)$ it must be absolutely continuous on $\delta_i(x_0)$. A like argument holds for almost all y_0.

3. *Invariance Properties*. The definition of the property a.c.s. seems highly artificial, since it is stated in terms of a particular coordinate system and has no self-evident invariance properties even under rotations of the axes. But this defect is only apparent; in fact, we shall now apply Theorem 1 to show that if a function is a.c.s. and has a finite Dirichlet integral, it retains these properties under all conformal transformations, the Dirichlet integral being unchanged in value. We shall prove the following theorem.

THEOREM 2. *Let the transformation*

$$x = x(x', y'), \ y = y(x', y') \tag{3}$$

map the bounded open set D' conformally on the bounded open set D and map $\bar{D}'$ topologically on $\bar{D}$, and let the function $v'(x', y')$ be a.c.s. on D' and have a finite Dirichlet integral

$$I'[v'] \equiv \iint_{D'} [v'^2_{x'} + v'^2_{y'}]dx'dy'.$$

Then its transform $v(x, y) = v'(x'(x, y), y'(x, y))$ is continuous on $\bar{D}$ and a.c.s. on D, and $I[v] = I'[v']$.

By drawing lines parallel to the x- and y-axes we subdivide the (x, y)-plane into squares of side 2^{-n}, where n is an arbitrary positive integer. We obtain the $(n+1)$th subdivision by subdividing each square of the nth. Those squares which are (with their boundaries) interior to D we call $q_1, q_2, \cdots, q_m$; the remainder of D is a set r, bounded in part by D^* and in part by line segments. In the q_j and in r we construct the harmonic functions which coincide with $v(x, y)$ on the boundaries of the subsets. Thus we have defined a function $u_n(x, y)$ which is continuous on D and harmonic on the interiors of the q_j and r. The transformation (3) carries $r, q_1, q_2, \cdots, q_m$ into subsets of D', which we call $r', q_1', q_2', \cdots, q_m'$, respectively; and it carries u_n' into a function $u_n'(x', y')$ which is continuous on D' and harmonic on the interior of each q_j' and r' and which coincides with v' on the boundary of each subset. Hence, by Theorem 1, the Dirichlet integral of v_n' over each subset is at most equal to the Dirichlet integral of u' over that set, so on adding these integrals we have $I'[u_n'] \leqq I'[v']$. But in each set q_i, r, the mapping (3) leaves the Dirichlet integral of the harmonic function u unchanged; hence $I[u_n]=I'[u_n']\leqq I'[v']$. Let us consider a point (x_0, y_0) of D. For all sufficiently large n, it belongs to some square, say $q_{i(n)}$, of the nth subdivision. Since the diameters of the squares approach zero, the greatest and least of the values of $v(x, y)$ on the boundary of $q_{i(n)}$ tend to $v(x_0, y_0)$; and since u_n is harmonic on the square, the value $u_n(x_0, y_0)$ lies between these two extreme values of v, so that $\lim u_n(x, y)=v(x,y)$ at each point of D. This equation continues to hold on D^*, where every u_n coincides with v. Finally, by §2, each $u_n(x, y)$ is a.c.s. on D.

We are now in a position to repeat the demonstration on pages 719–720 of my previously cited paper; we need only replace the word "uniformly" by "everywhere" and replace the interated integrals taken first with respect to v between $-(1-u^2)^{1/2}$ and $(1-u^2)^{1/2}$ and then with respect to v over CE, by iterated integrals taken first over $\sum\delta_i(x)$ and then over X. We thus find that $v(x, y)$ is a.c.s., and that

$$I[v] \leqq \liminf I[u_n] = \liminf I'[u_n'] \leqq I'[v'].$$

Interchanging the roles of v and v' shows that $I'[v']\leqq I[v]$; hence the Dirichlet integrals are equal, and the theorem is established. This theorem has an application in the theory of curved surfaces. Among all continuous surfaces

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \qquad ((u, v) \text{ on } B),$$

where B is a Jordan region, a particularly interesting class is that for which

(α) the functions $x(u, v)$, $y(u, v)$, $z(u, v)$ are a.c.s. on B,
(β) $E+G$ is summable, that is, $I[x]$, $I[y]$, $I[z]$ are finite.

In particular, if the region B can be taken to be a circle, α and β remaining satisfied, we say (with Morrey) that the surface is of type L_2. It is clearly more convenient to have a circle to deal with than to have a general Jordan region. But we can now show that every surface having a representation satisfying (α), (β) is of type L_2. We need only map B conformally on the unit circle; the transformed functions $\bar{x}(u, v)$, $\bar{y}(u, v)$, $\bar{z}(u, v)$, $u^2+v^2\leqq 1$, then satisfy (α) and (β) because of Theorem 2.

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