CONTINUED FRACTIONS AND CROSS-RATIO GROUPS OF CREMONA TRANSFORMATIONS*

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1. *Introduction.* An arbitrary cross ratio

$$r_{ijkl} = [z_i, z_j, z_k, z_l]$$

of four of the independent variables $z_1, z_2, \cdots, z_n$ is expressible rationally in terms of the ratios of any fundamental system such as

$$C: \qquad s_i = [z_{i-1}, z_{i-2}, z_i, z_{i-3}], \qquad (i = 4, 5, \cdots, n),$$

or

$$M: \qquad r_i = [z_1, z_2, z_3, z_i], \qquad (i = 4, 5, \cdots, n),$$

of $n-3$ independent ratios. If any particular system of $n-3$ independent ratios be associated with a particular order of the variables by varying the order of the n variables, we shall have in all $n!$ conjugate systems; these systems are expressible rationally in terms of the original system and in terms of any system of the set. Hence arises a group of $n!$ Cremona transformations on $n-3$ variables. E. H. Moore,† H. E. Slaught‡ and others have studied the group based on the initial system M.

In this paper I have shown that the transformations based on the system C have application to continued fractions of the form

$$\xi = 1 - \frac{x_1}{1} - \frac{x_2}{1} - \frac{x_3}{1} - \cdots.$$

If we take for the n variables $z_1, z_2, \cdots, z_n$ any n consecutive convergents of the continued fraction, say

$$(1) \qquad z_i = \frac{A_{q+i}}{B_{q+i}}, \qquad (i = 1, 2, \cdots, n; q \geqq -1),$$

* Presented to the Society, under a somewhat different title, April 6, 1934.

† E. H. Moore, American Journal of Mathematics, vol. 22 (1900), pp. 279–291.

‡ H. E. Slaught, ibid., pp. 343–380, and Part II in vol. 23.

then in this case the fundamental system C is $n-3$ consecutive elements x_i of the continued fraction ξ, namely,

$$s_i = x_{q+i}, \qquad (i = 4, 5, \cdots, n). \tag{2}$$

The Cremona transformations are on these elements, and their effect is to permute n consecutive convergents of ξ among themselves. The transformations do not disturb the convergence or divergence of the continued fraction, and result therefore in new convergence criteria.

2. *The Group $C_{n!}$ of Cremona Transformations.* Let $A_{m,t}/B_{m,t}$, $(t\geqq 0,\ A_{m,0}/B_{m,0}=A_m/B_m)$, be the mth convergent of the continued fraction

$$1 - \frac{x_{1+t}}{1} - \frac{x_{2+t}}{1} - \frac{x_{3+t}}{1} - \cdots.$$

Then with the aid of the identity*

$$A_{n+t-1}B_{t-1} - A_{t-1}B_{n+t-1} = -x_1x_2\cdots x_tB_{n-1,t},$$

we find that a cross ratio $r_{ijkl}=[z_i, z_j, z_k, z_l]$, $(i<j<k<l)$, of four of the n variables (1) can be expressed in the form

$$r_{ijkl} = \frac{B_{k-i-1,q+i+1}B_{l-j-1,q+j+1}}{B_{k-j-1,q+j+1}B_{l-i-1,q+i+1}}. \tag{3}$$

The other five distinct ratios of these four variables can be obtained from the ratio (3) by the well known transformations

$$\begin{gathered} r_{jikl} = \frac{1}{\lambda}, \qquad r_{ikjl} = 1-\lambda, \qquad r_{kijl} = \frac{1}{1-\lambda}, \\ r_{kjil} = \frac{\lambda}{\lambda-1}, \qquad r_{jkil} = \frac{\lambda-1}{\lambda}, \end{gathered} \tag{3a}$$

where $\lambda=r_{ijkl}$. In particular, we find that

$$s_i = x_{q+i}, \qquad (i = 4, 5, \cdots, n). \tag{4}$$

Inasmuch as (3) depends upon $x_{q+4}, x_{q+5}, \cdots, x_{q+n}$ only, it follows that (4) is a fundamental system of cross ratios.

Let $j_1, j_2, \cdots, j_n$ be an arbitrary permutation of $1, 2, \cdots, n$; and put

* Perron, *Die Lehre von den Kettenbrüchen*, 1st ed., p. 17.

$$a = \begin{pmatrix} 1, & 2, \cdots, & n \\ j_1, & i_2, \cdots, & i_n \end{pmatrix}.$$

Then

$$C^a: \qquad s_i^a = [z_{j_{i-1}}, z_{j_{i-2}}, z_{j_i}, z_{j_{i-3}}], \qquad (i = 4, 5, \cdots, n),$$

is also a fundamental system; and C^a is expressible rationally in terms of C:

$$s_i^a = f_i^a(s_4, s_5, \cdots, s_n), \qquad (i = 4, 5, \cdots, n),$$

or, simply, $C^a = f^a C$.

If b is another permutation, then $C^{ab} = f^a C^b = f^{ab} C$. The $n!$ Cremona transformations $f^a, f^b, \cdots$ form a group $C_{n!}$ simply isomorphic with the symmetric permutation group $G_{n!}$.

The ratios s_i are given in terms of the ratios r_i of E. H. Moore by the formula

$$(5) \qquad r_{ijkl} = [r_i, r_j, r_k, r_l], \qquad (r_1 = \infty, r_2 = 0, r_3 = 1).$$

The r_i can be expressed in terms of the s_i by means of (3) and (3a). Thus the group $C_{n!}$ is equivalent to that of E. H. Moore under the transformation

$$s_i = [r_{i-1}, r_{i-2}, r_i, r_{i-3}], \qquad (i = 4, 5, \cdots, n).$$

3. *Convergence Criteria for Continued Fractions.* Let $C_{m!}'$ be the subgroup of $C_{2m!}$ corresponding to permutations of the form

$$a = \begin{pmatrix} 1 & 2 & \cdots & m & 1+m & 2+m & \cdots & m+m \\ j_1 & j_2 & \cdots & j_m & j_1+m & j_2+m & \cdots & j_m+m \end{pmatrix},$$

in which $j_1, j_2, \cdots, j_m$ is a permutation of $1, 2, \cdots, m$. It is plain that $f_4^a, f_5^a, \cdots, f_m^a$ are functions of $s_4, s_5, \cdots, s_m$ alone, while $f_{m+4}^a, f_{m+5}^a, \cdots, f_{2m}^a$ are functions of $s_{m+4}, s_{m+5}, \cdots, s_{2m}$ alone. The remaining three functions depend in general upon $s_4, s_5, \cdots, s_{2m}$.

Let $k \geqq 0$, $m \geqq 3$, $z_{i+1} = A_{km+i}/B_{km+i}$, $(i = 0, 1, \cdots, 2m-1)$. Then $s_i = x_{km+i-1}$, $(i = 4, 5, \cdots, 2m)$. We have

$$x_{km+i-1}^a = f_i^a(x_{km+3}, x_{km+4}, \cdots, x_{(k+2)m-1}), \qquad (i = 4, 5, \cdots, 2m),$$

with the exceptional values*

* Perron, loc. cit., p. 198.

$$x_0^a = \frac{A_{j_1-1}}{B_{j_1-1}}, \qquad x_1^a = \frac{A_{j_1-1}}{B_{j_1-1}} - \frac{A_{j_2-1}}{B_{j_2-1}},$$

$$x_2^a = \left(\frac{A_{j_3-1}}{B_{j_3-1}} - \frac{A_{j_2-1}}{B_{j_2-1}}\right) \Big/ \left(\frac{A_{j_3-1}}{B_{j_3-1}} - \frac{A_{j_1-1}}{B_{j_1-1}}\right).$$

Then the continued fraction (with real or complex elements)

$$\xi_a = x_0^a - \frac{x_1^a}{1} - \frac{x_2^a}{1} - \frac{x_3^a}{1} - \cdots$$

has the same convergents as ξ, but in a different order. Consequently, if W_a is a region in m-space such that when the points

$$(x_{km}, x_{km+1}, \cdots, x_{(k+1)m-1}), \qquad (k = 0, 1, 2, \cdots),$$

range over W_a we shall always have*

$$| x_n^a | \leqq \frac{1}{4}, \qquad (n = 2, 3, 4, \cdots),$$

then ξ converges. In this manner every transformation of the group $C_{m!}'$ gives a convergence theorem for ξ.

4. *The Groups* C_{120} and C_{120}'. As an illustration,† we shall consider the groups $C_{5!}$ and $C_{5!}'$. The group C_{120} is generated by the four transformations

$$K \sim (34), \qquad L \sim (23)(45), \qquad M \sim (45), \qquad T \sim (12).$$

The three transformations K, L, M generate by themselves a subgroup C_{24} of the main group C_{120}; T will extend C_{24} to the main group.

By (3) and (3a) the transformations K, L, M, T are found to be as follows:

$$(6) \quad \begin{aligned} &K: x' = \frac{x}{x-1}, \; y' = \frac{1}{y}; \qquad L: x' = \frac{1-y}{x}, \; y' = y; \\ &M: x' = \frac{x}{1-y}, \; y' = \frac{y}{y-1}; \; T: x' = \frac{x}{x-1}, \; y' = \frac{y}{1-x}, \end{aligned}$$

where, to avoid subscripts, we have put $s_4 = x$, $s_5 = y$.

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* Perron, loc. cit., p. 259.

† The details of this illustration were worked out by Miss Lozelle Thomas.

In the xy-plane the geometrical configuration for the group C_{24} is as follows. The curves

1. $x^2=1-y$, 2. $y=0$, 3. $x=0$,

4. $y=(x-1)^2$, 5. $x=1$, 6. $y=1$,

7. $x+y=1$, 8. $x^2+2xy-2x-y+1=0$,

divide the plane into 24 regions. Take the fundamental region I to be the region in the second quadrant bounded by the curves 1, 2 and 3. The other regions in the second quadrant are then: L bounded by 1, 2, 6; LK bounded by 6, 8; LKL bounded by 7, 8; MKL bounded by 4, 7; and MK bounded by 3, 4.

In the first quadrant the regions are: K bounded by 3, 8; KL bounded by 5, 6, 8; $KMKL$ bounded by 4, 5, 6, extending to infinity; MLK bounded by 4, 6; $MLKM$ bounded by 2, 4, 6; $MLKLM$ bounded by 4, 5, 6; KLK bounded by 1, 5, 6; $LKLK$ bounded by 1, 7; $MKLM$ bounded by 4, 7; and KMK bounded by 2, 3, 4.

In the third quadrant the regions are: LM bounded by 1, 2; and M bounded by 1, 2, 3.

In the fourth quadrant the regions are: KM bounded by 2, 3, 8; KLM bounded by 5, 8; $MLKLK$ bounded by 1, 5; $MKLK$ bounded by 1, 7; $LKLM$ bounded by 7, 8; and LKM bounded by 2, 8.

The generators of C_{120} correspond to the permutations (34) (89), (23)(45)(78)(9, 10), (45)(9, 10), and (12)(67). If we denote them by K', L', M', T', respectively, then these transformations are found to be

$$K': \quad \begin{aligned} x' &= \frac{x}{x-1}, \quad y' = \frac{1}{y}, \quad \bar{x}' = \frac{\bar{x}}{\bar{x}-1}, \quad \bar{y}' = \frac{1}{\bar{y}}, \\ u' &= \frac{u}{u-1}, \quad v' = \frac{v}{1-w}, \quad w' = \frac{w}{1-\bar{x}}; \end{aligned}$$

$$L': \quad \begin{aligned} x' &= \frac{1-y}{x}, \quad y' = y, \quad \bar{x}' = \frac{1-\bar{y}}{\bar{x}}, \quad \bar{y}' = \bar{y}, \\ u' &= \frac{1-y}{u}, \quad v' = \frac{v}{v+w-1}, \quad w' = \frac{w}{v+w-1}; \end{aligned}$$

$$M': \quad x' = \frac{x}{1-y}, \quad y' = \frac{y}{y-1}, \quad \bar{x}' = \frac{\bar{x}}{1-\bar{y}}, \quad \bar{y}' = \frac{\bar{y}}{\bar{y}-1}, \quad u' = \frac{1}{u}, \quad v' = \frac{v}{v-1}, \quad w' = \frac{w}{1-v};$$

$$T': \quad x' = \frac{x}{x-1}, \quad y' = \frac{y}{1-x}, \quad \bar{x}' = \frac{\bar{x}}{\bar{x}-1}, \quad \bar{y}' = \frac{\bar{y}}{1-\bar{x}}, \quad u' = \frac{u}{1-v}, \quad v' = \frac{v}{v-1}, \quad w' = \frac{1}{w},$$

where for simplicity we have put $x = x_{5m+3}$, $y = x_{5m+4}$, $u = x_{5m+5}$, $v = x_{5m+6}$, $w = x_{5m+7}$, $\bar{x} = x_{5m+8}$, $\bar{y} = x_{5m+9}$.

Each of these transformations yields a convergence theorem for the continued fraction ξ in accordance with the remarks in §3. For example, K' gives the following theorem.

The continued fraction ξ converges if the following inequalities hold:

$$\left|\frac{x_2}{1-x_3}\right| \leqq \frac{1}{4}; \quad \left|\frac{x_{5n+3}}{x_{5n+3}-1}\right| \leqq \frac{1}{4}, \quad \left|\frac{1}{x_{5n+4}}\right| \leqq \frac{1}{4}, \quad (n = 0, 1, 2, \cdots),$$

$$\left|\frac{x_{5n}}{x_{5n}-1}\right| \leqq \frac{1}{4}, \quad \left|\frac{x_{5n+1}}{1-x_{5n+2}}\right| \leqq \frac{1}{4}, \quad \left|\frac{x_{5n+2}}{1-x_{5n+3}}\right| \leqq \frac{1}{4}, \quad (n = 1, 2, 3, \cdots),$$

where $x_1, x_2, x_3, \cdots$ are real or complex numbers.

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