# ON THE FINITENESS OF THE CLASS NUMBER IN A SEMI-SIMPLE ALGEBRA*

BY C. G. LATIMER

1. *Introduction.* Let $A$ be a rational semi-simple algebra of order $n$ and let $G$ be a domain of integrity of order $n$ in $A$, according to Dickson's definition.† In a recent paper,‡ Miss Shover showed that if $A$ is a division algebra, the number of classes of left ideals in $G$ is finite. She used the definitions of an ideal and a class of ideals as given by MacDuffee. § We shall extend this result to any $A$. Since Miss Shover also showed that there is a one-to-one correspondence between the classes of left ideals and the classes of right ideals in $G$, it will be sufficient to prove the theorem for right ideals. By applying this result, we shall obtain a theorem on similar matrices.

Artin proved the finiteness of the right ideal class number for a maximal ordnung in $A$, using a different definition of an ideal and a class of ideals.‖ Every domain of integrity of order $n$ is an ordnung. An ordnung is a domain of integrity of order $n$ if and only if it contains the modulus of $A$. In particular, every maximal ordnung is a domain of integrity of order $n$. Every non-singular ideal according to MacDuffee is an ideal according to Artin and if $\beta$ is an element and $K$ is an ideal in $G$, MacDuffee's and Artin's definitions of their norms $N(\beta)$, $N(K)$ are the same.

2. *Proof of the Theorem.* The only place in her paper where Miss Shover employed the hypothesis that $A$ is a division algebra was in obtaining, for left ideals, a result equivalent to the following, which was proved by Artin for the case where $G$ is maximal.

LEMMA 1. *There is a positive number $C$, depending only on $G$, such that if $K$ is a non-singular right ideal in $G$, there is an element $\beta$ in $K$ for which*

* Presented to the Society, December 27, 1933.

† *Algebren und ihre Zahlentheorie*, p. 155.

‡ This Bulletin, vol. 39 (1933), pp. 610–14.

§ Transactions of this Society, vol. 31 (1929), pp. 71–90.

‖ Abhandlungen, Mathematisches Seminar, Hamburg, vol. 5 (1927), pp. 251–289.

$$0 < |N(\beta)| \leqq CN(K).$$

There is a maximal domain $G_0$ containing $G$.* Consider the following lemma.

LEMMA 2. *Lemma* 1 *is valid if it is valid when* $G$ *is replaced by* $G_0$.

This lemma was proved by Artin, using a different notation, for the case where $G$ and $G_0$ are both maximal.† However, the present lemma may be proved by exactly the same argument as that made by Artin.

But by Artin's Theorem 17 (p. 283), our Lemma 1 is valid for $G_0$. Lemma 1 follows.

Let $K$ be a non-singular right ideal in $G$ and let $\beta$ be an element in $K$ satisfying the conditions of Lemma 1. Following Miss Shover's proof, we find that the transpose of the first matrix of $\beta$ is $R(\beta)=MG$, where $G$ is the matrix of $K$ and $M$ is a matrix with integral elements. Noting that $|R(\beta)|=N(\beta)\neq 0$ and letting $\pm|M|=m>0$, we have

$$0 < |N(\beta)| = \pm|M| \cdot |G| = m\cdot N(K) \leqq C\cdot N(K).$$

Furthermore, the adjoint of $M$ is the matrix of an ideal $L$, in the same class as $K$. Then by the last of the above inequalities, we have

$$0 < N(L) = m^{n-1} \leqq C^{n-1}.$$

Hence every class of ideals contains a non-singular ideal with norm $\leqq C^{n-1}$. Since there is only a finite number of ideals with norms equal to a given positive integer, we have the following theorem.

THEOREM 1. *The number of classes of right ideals in* $G$ *is finite.*

3. *An Application of Theorem* 1. It will be understood that all matrices referred to are square matrices of order $n$ with integral elements. Two matrices, $A$ and $A_1$, are said to be similar if there is a unimodular matrix $Z$, such that $A_1=ZAZ^{-1}$. All matrices similar to the same matrix are said to form a class. Two

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* Artin, loc. cit., p. 265.

† Loc. cit., pp. 283–284.

matrices are similar if and only if they belong to the same class. Let

$$f(x) = x^n + k_1x^{n-1} + \cdots + k_n,$$

where the $k$'s are rational integers, $k_n \neq 0$, and $f(x) = 0$ has no multiple roots. If $A$ is a matric root of $f(x) = 0$ and is non-derogatory, that is, is not a root of an equation, with rational coefficients, of lower degree, the same is true of every matrix similar to $A$. It is known that there is a one-to-one correspondence between the classes of ideals in a domain of integrity in a certain commutative semi-simple algebra and the classes of non-derogatory matrices which are roots of $f(x) = 0$.* We have therefore, by Theorem 1, the following result.

THEOREM 2. *The number of classes of non-derogatory similar matrices which are roots of* $f(x) = 0$ *is finite.*

THE UNIVERSITY OF KENTUCKY

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# KASNER'S CONVEX CURVES†

BY MORRIS HALPERIN

1. *Preliminary Discussion.* A Kasner convex curve is the limit of a sequence of simple, closed, convex polygons, $P_0, \cdots, P_n, \cdots$, each of which has a finite number of sides and is obtained from the preceding one by measuring off the $r$th part of the length of each side from both its ends and cutting off the corners. The number $r$ is restricted to the inequality $0 < r < 1/2$. To obtain an analytic definition for the curve, we proceed as follows. We note that the centroid of the vertices of $P_0$ is also the centroid of the vertices of every $P_n$. Hence $G$ is interior to every $P_n$. Let $z_n(t)$ be the intersection of a ray from $G$ of inclination $t$ with the polygon $P_n$. The sequence of functions $\{z_n(t)\}$ will be found to converge uniformly to a function $z(t)$.

* Latimer and MacDuffee, *A correspondence between classes of ideals and classes of matrices*, Annals of Mathematics, (2), vol. 34 (1933), pp. 313–316.

† Presented to the Society, February 25, 1933. Another paper will follow in which additional properties of these curves will be discussed; particularly their second derivatives, their non-analytic character, and their areas. See this Bulletin, Abstract 39-3-68.