ON CONVERGENCE IN VARIATION*

BY C. R. ADAMS AND J. A. CLARKSON

1. *Introduction.* Certain questions concerning functions $f(x, y)$ of bounded variation naturally lead one to consider a sequence of functions $f_n(x)$, $(n=1, 2, 3, \cdots)$, defined on an interval† (a, b) and satisfying the following conditions: $f_n(x)$ tends to a limit function $f_0(x)$ of bounded variation; the total variation $T_a^b(f_n)$ of $f_n(x)$ on (a, b) tends to the total variation $T_a^b(f_0)$ of $f_0(x)$ on (a, b).‡ The notation $f_n(x)$—v→$f_0(x)$ will frequently be employed to describe this situation, which has already received attention from Buchanan and Hildebrandt.§ All of the theorems which we are about to establish are valid when a set of functions $f(x, \lambda)$ corresponding to a set of values λ having λ_0 as a limit is considered, with $f(x, \lambda)\rightarrow f_0(x)$ as $\lambda\rightarrow\lambda_0$ over the set.

2. *Preliminary Theorems.* Let $P_n[N_n]$ denote the total positive [negative] variation of $f_n(x)$ on (a, b), $(n=0, 1, 2, \cdots)$; then we have the following theorem.

THEOREM 1. *The relations* $f_n(a)\rightarrow f_0(a)$, $f_n(b)\rightarrow f_0(b)$, *and* $T_a^b(f_n)\rightarrow T_a^b(f_0)$ *imply* $P_n\rightarrow P_0$ *and* $N_n\rightarrow N_0$.

This follows at once by writing

$$f_n(b)=f_n(a)+P_n-N_n, \qquad (n=0, 1, 2, \cdots).$$

THEOREM 2. *The relation* $f_n(x)\rightarrow f_0(x)$ *on* (a, b) *implies*

$$\liminf_{n\rightarrow\infty} T_a^b(f_n) \geqq T_a^b(f).$$

This may easily be proved directly or by aid of the well

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† The *closed* interval is always to be understood.

‡ It may be of interest to note that $T_a^b(f(x))$ is a *semi-linear* operation in the sense that we have $T_a^b(f(x)+g(x))\leqq T_a^b(f(x))+T_a^b(g(x))$ and $T_a^b(cf(x)) =|c|\, T_a^b(f(x))$ for c constant.

§ Buchanan and Hildebrandt, *Note on the convergence of a sequence of functions of a certain type*, Annals of Mathematics, (2), vol. 9 (1908), pp. 123–126. This paper will be referred to as BH. The symbol —v→ may be read "*converges in variation.*"

known fact that if f_n tends to f_0 on (a, b) and $T_a^b(f_n)$ is $\leqq M$, $T_a^b(f_0)$ is $\leqq M$.

COROLLARY. *The relation* $f_n(x)$—v→$f_0(x)$ *on* (a, b) *implies that relation for every subinterval.*

THEOREM 3. *If we have* $f_n(x)\to f_0(x)$ *on a set of points everywhere dense in* (a, b), *with* $T_a^b(f_n)\to T_a^b(f_0)$ *and* $f_0(x)$ *continuous on* (a, b), $f_n(x)$ *tends to* $f_0(x)$ *everywhere on* (a, b).

This is sufficiently clear.

3. *Uniform Convergence.* We have the following theorems.

THEOREM 4. *The relations* $f_n(x)$—v→$f_0(x)$ *on* (a, b) *and* $f_0(x') = f_0(x'-0)$ $[f_0(x') = f_0(x'+0)]$ *imply that* x' *is a point of uniform convergence on the left* [*right*] *for both* $f_n(x)$ *and* $T_a^x(f_n)$.*

PROOF. $T_a^x(f_0)$ is continuous on the left at x'; any $\epsilon>0$ being given, choose $\delta(>0)$ so that $T_{x'-\delta}^{x'}(f_0)$ is $<\epsilon$ and then m so that we have for $n>m$

$$|f_n(x') - f_0(x')| < \epsilon, \qquad |T_a^{x'-\delta}(f_n) - T_a^{x'-\delta}(f_0)| < \epsilon,$$
$$T_{x'-\delta}^{x'}(f_n) < 2\epsilon.$$

Then we have $|f_n(x)-f_0(x)|<4\epsilon$ and $|T_a^x(f_n)-T_a^x(f_0)|<4\epsilon$ for $0\leqq x'-x\leqq\delta$, $n>m$.

COROLLARY. *The hypotheses of Theorem* 3 *imply that the convergence of both* $f_n(x)$ *to* $f_0(x)$ *and* $T_a^x(f_n)$ *to* $T_a^x(f_0)$ *is uniform over* (a, b).†

THEOREM 5. *The relations* $f_n(x)$—v→$f_0(x)$ *on* (a, b) *and* $f_0(x') \neq f_0(x'-0)$ $[f_0(x') \neq f_0(x'+0)]$ *imply that* x' *is a point of uniform convergence on the left* [*right*] *for both* $f_n(x)$ *and* $T_a^x(f_n)$ *or for neither, according as* $f_n(x'-0)$ $[f_n(x'+0)]$ *tends to* $f_0(x'-0)$ $[f_0(x'+0)]$ *or not.*

PROOF. Let $\bar{f}_n(x) = f_n(x)$ for $a\leqq x<x'$, $\bar{f}_n(x') = f_n(x'-0)$, $(n=0, 1, 2, \cdots)$; then we have $T_a^x(f_n) = T_a^x(\bar{f}_n) + |f_n(x)-\bar{f}_n(x)|$ for each n and $a\leqq x\leqq x'$. If x' is a point of uniform convergence on the left for *either* $f_n(x)$ or $T_a^x(f_n)$, we have $\bar{f}_n(x')\to\bar{f}_0(x')$. By Theorem 4, x' is then a point of uniform convergence on the

* The part of this theorem concerning convergence of f_n is not essentially different from the Lemma of BH; the same is true of the proofs.

† This result includes Theorem B of BH on uniform convergence of f_n.

left for *both* $\bar{f}_n(x)$ and $T_a^x(\bar{f}_n)$, and hence for *both* $f_n(x)$ and $T_a^x(f_n)$.

COROLLARY. *If we have* $f_n(x)—v\rightarrow f_0(x)$ *on* (a, b), *a necessary and sufficient condition that* $T_a^x(f_n)\rightarrow T_a^x(f_0)$ *uniformly on* (a, b) *is that* $f_n(x)\rightarrow f_0(x)$ *uniformly on* (a, b).

It may be observed that when $T_a^x(f_n)$ converges uniformly, the same is true of the total positive and negative variations, $P_a^x(f_n)$ and $N_a^x(f_n)$.

4. *Reciprocal Sequences.* Let S be any set of points x_i, $(i=0, 1, \cdots, p)$, with $a=x_0<x_1<\cdots<x_p=b$, and let $\sum(f, S) =\sum_{i=1}^{p}|f(x_i)-f(x_{i-1})|$. For the proof of our next theorem the following rather obvious lemma is convenient.

LEMMA. *Let* $f(x)$ *be finite and* $>\alpha>0$ (*or* $<-\alpha<0$) *on* (a, b), *and let* S' *be the set obtained by adding a new point* X *to* S; *then we have*

$$0 \leqq \sum(1/f, S') - \sum(1/f, S) \leqq [\sum(f, S') - \sum(f, S)]/\alpha^2.$$

THEOREM 6. *The relation* $f_n(x)—v\rightarrow f_0(x)$ *on* (a, b), *when* $|f(x)|$ *is* $>2\alpha>0$ *and* $f(x)$ *does not change sign in the interval, implies* $1/f_n(x)—v\rightarrow 1/f_0(x)$.

PROOF. It may readily be shown that for n sufficiently large the functions f_n are uniformly $>\alpha$ (or $<-\alpha$). Thus no loss of generality results from assuming, as we now do, that the sequence f_n is uniformly $>\alpha$ and is of uniformly bounded variation. Then there exists a double sequence of sets $S_n^{(p)}$, $(p, n=0, 1, 2, \cdots)$, such that for each p we have

$$S_{n+1}^{(p)} \geqq S_n^{(p)}, \qquad S_n^{(p+1)} \geqq S_n^{(p)}, \quad (n = 0, 1, 2, \cdots);$$

$$\sum(f_p, S_n^{(p)}) \rightarrow T(f_p), \qquad \sum(1/f_p, S_n^{(p)}) \rightarrow T(1/f_p)$$

as $n\rightarrow\infty$. Let $S_n=S_n^{(n)}$ and consider the double sequences $a_{mn}=\sum(f_m, S_n)$, $b_{mn}=\sum(1/f_m, S_n)$. Because of our choice of sets and the relation $f_n—v\rightarrow f_0$, a_{mn} is non-decreasing in n for each m and the iterated limits as $m, n\rightarrow\infty$ both exist and are equal. By a Lemma of Hildebrandt,* $\lim_n a_{mn}$ then exists uniformly in m. From the above Lemma we infer that $\lim_n b_{mn}$ exists uniformly

* Hildebrandt, *On a generalization of a theorem of Dini on sequences of continuous functions*, this Bulletin, vol. 21 (1914), pp. 113–115.

in m, and by our choice of sets we have $\lim_m b_{mn} = \sum(1/f, S_n)$; hence by a classical theorem we have

$$\lim_n \lim_m b_{mn} = T(1/f) = \lim_m \lim_n b_{mn} = \lim_m T(1/f_m).$$

That Theorem 5 *is not true when the hypothesis that* $f(x)$ *be of fixed sign in* (a, b) *is deleted* may readily be seen. It should be remarked first that if f changes sign in (a, b), the fact that $|f|$ is bounded away from zero does not imply that $f_n(x)$, for n sufficiently large, is uniformly bounded away from zero; hence it would be natural to make the latter condition a part of the hypotheses in considering this case. But even then the theorem would be untrue, as is seen from such a simple example as the following: $f_0(x) = 1$ for $0 \leqq x < 1$, $f_0(1) = -1$; $f_n(x)$, $(n = 1, 2, 3, \cdots)$, a monotone function decreasing from 1 to -1 and continuous except for a jump from $1/2$ to $-1/2$ somewhere in the interval, with $f_n \rightarrow f_0$.

The following application to functions of two variables may be noted. Let $f(x, y)$ be defined over a rectangle $R(a \leqq x \leqq b, c \leqq y \leqq d)$, and let $\phi(\bar{x})$ stand for the total variation of $f(\bar{x}, y)$ in y over the interval $c \leqq y \leqq d$; then we have the following fact.

COROLLARY. *If* $f(x, y)$ *is* $> \alpha > 0$ (*or* $< -\alpha < 0$) *in* R *and is continuous in* x, $\phi(x)$ *for* $1/f(x, y)$ *is continuous wherever* $\phi(x)$ *for* $f(x, y)$ *is continuous.*

5. *Sums and Products of Sequences.* One may readily show that the relations $f_n(x) \text{—}v\text{→} f_0(x)$ and $g_n(x) \text{—}v\text{→} g_0(x)$ on (a, b), even when $f_0(x)$ and $g_0(x)$ are both continuous on (a, b) (which by the corollary to Theorem 4 provides all the uniformity of convergence that could be desired), imply neither $f_n + g_n \text{—}v\text{→} f_0 + g_0$ nor $f_n g_n \text{—}v\text{→} f_0 g_0$. In fact we can make the stronger assertion that $f_n \text{—}v\text{→} f_0$, with g_0 of bounded variation, implies neither $f_n + g_0 \text{—}v\text{→} f_0 + g_0$ nor $f_n g_0 \text{—}v\text{→} f_0 g_0$. The following example exhibits a sequence of absolutely continuous* functions f_n converging in variation to an absolutely continuous* limit function f_0, and an absolutely continuous* function g_0, for which $f_n + g_0$ does not converge in variation to $f_0 + g_0$: for $0 \leqq x \leqq 1$, let $g_0(x) = -x$ and

$$f_n(m/2^n) = \begin{cases} m/2^n & \text{for } m \equiv 0 \bmod 2, \\ m/2^n + 1/2^{n+1} & \text{for } m \equiv 1 \bmod 4 \\ m/2^n - 1/2^{n+1} & \text{for } m \equiv 3 \bmod 4, \ (n = 1, 2, 3, \cdots), \end{cases}$$

* The functions f_0, g_0, and f_n, $(n = 1, 2, 3, \cdots)$, are all monotone.

and between the points $m/2^n$, let f_n be defined linearly.

6. *A Set of Conditions Sufficient to Insure Convergence in Variation.*

THEOREM 7. *Let $f_n(x)$ be a sequence of absolutely continuous functions converging to a limit function $f_0(x)$ on (a, b); let $f_n'(x)$ converge asymptotically to a limit function, and let $f_n'(x)$, $(n=1, 2, 3, \cdots)$, be dominated by a summable function; then we have $f_n(x)$—v→$f_0(x)$ on (a, b).*

It is easily seen that the hypotheses imply (i) that $f_0(x)$ is absolutely continuous, so that we may write

$$T_a{}^b(f_n) = \int_a^b |f_n'(x)| dx, \quad (n = 0, 1, 2, \cdots),$$

and (ii) that we may pass to the limit under the integral sign.

COROLLARY. *Let the series $\sum_{i=0}^{\infty} a_i x^i$, with real coefficients, have the radius of convergence $R(>0)$; let the sum of the series be denoted by $S(x)$, and let $S_n(x) = \sum_{i=0}^{n} a_i x^i$; then we have $S_n(x)$ —v→$S(x)$ on each interval (a, b), $(-R<a<b<R)$.*

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TYPES OF INVOLUTORIAL SPACE TRANSFORMATIONS ASSOCIATED WITH CERTAIN RATIONAL CURVES—COMPOSITE BASIS CURVES*

BY AMOS BLACK

1. *Introduction.* In a preceding paper† the author found and discussed the involutorial transformations belonging to the special complex of lines which meet a rational curve r of order m, $(m=2, 3, 4, 5)$, and having a pencil of invariant cubic surfaces which contain the curve r as a simple basis element, with the restriction that the residual basis curve, γ_{9-m}, of the pencil should not be composite. In this paper we shall discuss the cases where γ_{9-m} is composite.

2. *Equations of the Transformation.* The equations of the

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† *Types of involutorial space transformations associated with certain rational curves*, Transactions of this Society, vol. 34 (1932), pp. 795–810.