# SOME DEFINITE INTEGRALS INVOLVING SELF-RECIPROCAL FUNCTIONS

BY BRIJ MOHAN MEHROTRA

1. *Introduction.* In one of his papers, Ramanujan* has proved formally that if

$$\phi_\omega(t) = \int_0^\infty \frac{\cos \pi t x}{\cosh \pi x} e^{-\pi\omega x^2} dx,$$

then

$$\phi_\omega(t) = \frac{e^{-\pi t^2/(4\omega)}}{\omega^{1/2}} \phi_{1/\omega}\left(\frac{it}{\omega}\right). \tag{1}$$

An examination of the proof shows that it rests on the fact that $\operatorname{sech}[x(\pi/2)^{1/2}]$ is self-reciprocal for cosine-transforms. The present investigation was suggested by this fact. The object of this note is to obtain a generalization of (1).

Following Hardy and Titchmarsh, I will say that a function is $R_\nu$ if it is its own $J_\nu$ transform, and it is $-R_\nu$ if it is skew-reciprocal for $J_\nu$ transforms; also, for $R_{1/2}$ and $R_{-1/2}$, I will write $R_s$ and $R_c$, respectively.

2. THEOREM 1. *If*

$$\phi_\omega(t) = \omega^{1/2} \int_0^\infty e^{-\omega^2 x^2/2} f(x) \cos t\omega x \, dx,$$

*where $f(x)$ is $R_c$ and is such that $\int_0^\infty |f(x)|\, dx$ converges, then*

$$\phi_\omega(t) = e^{-t^2/2} \phi_{1/\omega}(it). \tag{2}$$

We have

$$\phi_\omega(t) = \left(\frac{2\omega}{\pi}\right)^{1/2} \int_0^\infty e^{-\omega^2 x^2/2} \cos t\omega x \, dx \int_0^\infty f(y) \cos xy \, dy. \tag{3}$$

This double integral is absolutely convergent, as we see by comparison with

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* Ramanujan, *Some definite integrals*, Collected Papers, Cambridge University Press, 1927, pp. 202–207.

$$\int_0^\infty e^{-\omega^2 x^2/2} dx \int_0^\infty |f(y)|\, dy.$$

Hence we may invert the order of integration in (3). Thus

$$\begin{aligned}
\phi_\omega(t) &= \left(\frac{2\omega}{\pi}\right)^{1/2} \int_0^\infty f(y)dy \int_0^\infty e^{-\omega^2 x^2/2} \cos t\omega x \cos xy\, dx \\
&= \left(\frac{\omega}{2\pi}\right)^{1/2} \int_0^\infty f(y)dy \int_0^\infty e^{-\omega^2 x^2/2} \{\cos (y + t\omega)x \\
&\quad + \cos (y - t\omega)x\} dx \\
&= \frac{1}{2\omega^{1/2}} \int_0^\infty f(y) \{ e^{-(y+t\omega)^2/(2\omega^2)} + e^{-(y-t\omega)^2/(2\omega^2)} \} dy \\
&= \frac{1}{2\omega^{1/2}} \int_0^\infty e^{-y^2/(2\omega)^2 - t^2/2} (e^{-yt/\omega} + e^{yt/\omega}) f(y) dy \\
&= \frac{e^{-t^2/2}}{\omega^{1/2}} \int_0^\infty e^{-y^2/(2\omega^2)} \cosh \frac{yt}{\omega} f(y) dy,
\end{aligned}$$

which establishes (2). As an illustration, (2) may be verified for $f(x) = e^{-x^2/2}$.

3. THEOREM 2. *If*

$$\psi_\omega(t) = \omega^{1/2} \int_0^\infty e^{-\omega^2 x^2/2} f(x) \sin t\omega x\, dx,$$

*where* $f(x)$ *is* $R_s$ *and is such that* $\int_0^\infty |f(x)|\, dx$ *converges, then*

$$\psi_\omega(t) = - i e^{-t^2/2} \psi_{1/\omega}(it).\dagger \tag{4}$$

This can be proved in exactly the same way as Theorem 1. To illustrate this theorem, (4) may be verified for $f(x) = xe^{-x^2/2}$.

MORADABAD, INDIA

† Theorems 1 and 2 themselves depend upon the fact that $e^{-x^2/2}$ is $R_c$, and can be further generalized, but the generalized theorems do not seem to be very useful.