VEBLEN ON PROJECTIVE RELATIVITY

Projektive Relativitätstheorie. By O. Veblen. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 2, Heft 1. Berlin, Springer, 1933. 6+73 pp.

This book by Professor Veblen is a result of a series of lectures given at the University of Göttingen during the summer of 1932. It deals with that new aspect of the theory of relativity which is often called projective relativity on account of its relation to projective geometry. It allows a unified theory of the gravitational and the electromagnetic field, and, though it is not referred to in Veblen's book, also offers a possibility of including modern wave mechanics. Apart from this it throws a new light on such an old theory as classical projective geometry. The material is mainly taken from papers by the author himself and by close collaborators. The exposition is elegant and clear.

The present theory is therefore a result of two series of investigations, one mathematical and one physical. The mathematical side is the theory of projective connections, the physical side consists in the many attempts, begun by H. Weyl in 1918, to define a space-time structure depending not only on gravitational but also on electromagnetic potentials. Projective relativity seems to offer a rather simple and attractive solution.

The theory of projective connections is a generalization of projective geometry in the same sense as Riemannian geometry is a generalization of euclidean geometry. It is a theory of manifolds for which ordinary projective relations exist in the immediate neighborhood of a generating point, these relations being connected by a law which makes the manifold a "curved" projective manifold.

This is done in the following way. In a four-dimensional manifold with coordinates x^i, $i=1, 2, 3, 4$, there belongs to every point a "tangential space" of the dx^i which can be considered as an affine space. In a Riemannian geometry of fundamental tensor g_{ij} we have, at every point, a "light cone" $g_{ij}dx^i dx^j=0$ in the tangential space. We now introduce a non-degenerate quadric of which this cone is the asymptotic cone, and it is possible to define in each tangential space a non-euclidean geometry with respect to this quadric. The projective differential geometry of this kind of relativity is the "curved" generalization of this geometry.

To master the properties homogeneous coordinates are introduced into the space of the dx^i through the relations

$$dx^i = X^i/(\phi_\alpha X^\alpha), \qquad (i = 1, 2, 3, 4; \alpha = 0, 1, 2, 3, 4);$$

the ϕ_α are functions which allow us to write for the hyperplane at infinity the equation $\phi_\alpha X^\alpha=0$. Homogeneous coordinates do not change when they are multiplied by a factor e^{x^0} of the coordinates (x^1, x^2, x^3, x^4). With respect to the transformations

$$\bar{x}^0 = x^0 + \log \rho, \quad \bar{x}^i = \bar{x}^i(x), \; (\rho \text{ a function of } x^1, \cdots, x^4)$$

we can then define "projective tensors," for example,

$$\bar{G}_{\alpha\beta} = G_{\sigma\tau} \frac{\partial x^\sigma}{\partial \bar{x}^\alpha} \frac{\partial x^\tau}{\partial \bar{x}^\beta}, \qquad (\alpha, \beta = 0, 1, 2, 3, 4).$$

From the equations $X^i = kdx^i$, $X^0 = k(1 - \phi_i dx^i)$, where k is an arbitrary number, the transformations of the X^α can be found:

$$\bar{X}^i = Kd\bar{x}^i = \frac{K}{k} X^j \frac{\partial \bar{x}^i}{\partial x^j}, \quad \bar{X}^0 = K(1 - \bar{\phi}_i d\bar{x}^i) = K(1 - \phi_i dx^i) = \frac{K}{k} X^0,$$

where K is a proportionality factor.

In the tangential space the equation $G_{\alpha\beta}X^\alpha X^\beta = 0$ determines a quadric, which can be used for the determination of a non-euclidean metric. The tensor $G_{\alpha\beta}$ "contains" a projective scalar ϕ, a projective vector ϕ_α and an affine tensor g_{ij}, according to the formula

$$G_{\alpha\beta} = \phi^2(g_{\alpha\beta} + \phi_\alpha \phi_\beta) = \phi^2 \gamma_{\alpha\beta},$$
$$G_{00} = \phi^2, \quad G_{x\beta}/G_{00} = \gamma_{\alpha\beta}, \quad G_{0\alpha}/G_{00} = \phi_\alpha.$$

The g_{ij} and ϕ_α can be used to determine gravitational and electromagnetic potentials.

The next task is to relate the local spaces by means of a connection. First it is shown how classical projective geometry can be obtained. Here exist preferred coordinate systems Z^α, "projective coordinate systems," connected by transformation formulas of the kind $\bar{Z}^\alpha = p^\alpha Z^\beta$ which are related to the x^i by means of the equations $Z^\alpha = e^{x^0} f^\alpha(x^1, x^2, x^3, x^4)$. Classical projective geometry is now characterized by the existence of a family of projective scalars $Z = p_\alpha A^\alpha$ with arbitrary constants p_α. Elimination of the p_α leads to the differential equations

$$\frac{\partial^2 Z}{\partial x^\alpha \partial x^\beta} - \Pi^\sigma_{\alpha\beta} \frac{\partial Z}{\partial x^\sigma} = 0, \qquad \frac{\partial Z}{\partial x^0} = Z,$$

for which the author suggests the name of "differential equations of projective geometry." The functions $\Pi^\sigma_{\alpha\beta}$ transform under a transformation of the x^0, x^i as the parameters of connection of an affine five-dimensional manifold:

$$\bar{\Pi}^\alpha_{\beta\gamma} = \left(\Pi^\sigma_{\rho\tau} \frac{\partial x^\rho}{\partial \bar{x}^\beta} \frac{\partial x^\tau}{\partial \bar{x}^\gamma} + \frac{\partial^2 x^\sigma}{\partial \bar{x}^\beta \partial \bar{x}^\gamma} \right) \frac{\partial \bar{x}^\alpha}{\partial x^\beta}.$$

The $\Pi^\alpha_{\beta\gamma}$ define a special type of projective connection.

The differential equations for Z are not the most general of their kind, because they satisfy special integrability conditions. Their curvature tensor $R^\lambda_{\alpha\beta\gamma}$ must vanish. If we discard these integrability conditions, we obtain the general "curved" projective geometry, also with parameters of connection $\Pi^\alpha_{\beta\gamma}$. The author defines this in detail, and also shows how the transition to non-homogeneous coordinates can be made. He then shows how infinitesimal projective displacements can be determined, where the $\Lambda^\sigma_{\alpha\beta}$ depend on $\Pi^\sigma_{\alpha\beta}$ (but not uniquely):

$$\frac{\partial X^\alpha}{\partial x^\sigma} + \Lambda^{\ \alpha}_{\beta\sigma} X^\beta = 0.$$

It is important to observe that in projective differential geometry connection ("Zusammenhang") and displacement ("Uebertragung") must be separately treated. The connection determines only under certain circumstances a displacement. There is also a difficulty with the generalization of the notion of path, as the coordinates X^α determine a point, not a vector. Paths are here defined as curves for which points in the tangential space pass by displacement into themselves.

The projective connection and its displacements can now be made to depend on $G_{\alpha\beta}$. A special case is the non-euclidean geometry of the Cayley type, which finds extensive description. Generalization to the theory of a general projective tensor of second order leads to a "generalized theory of conic sections," which not only contains a Riemannian geometry, but also contains non-metrical properties. This means in the physical interpretation that gravitational and electromagnetic elements are possible.

The geometry is now ready for physical interpretation. The "paths" of the projective displacement defined by $G_{\alpha\beta}$ become the world lines of an electrical particle; they also include the geodesic lines of the metric. A variation principle

$$\delta \int B g^{1/2} dx^1 dx^2 dx^3 dx^4 = 0,$$

where B is the curvature scalar belonging to $\gamma_{\alpha\beta}$ and the variation is carried out with respect to $\gamma_{\alpha\beta}$ with $\gamma_{00} = 1$, gives the field equations for empty space. When they are split into their affine parts they yield the ordinary field equations of relativity. The physical theory only uses the g_{ij}, ϕ_i parts of $G_{\alpha\beta}$, not the ϕ. The author remarks that it may be possible to connect this ϕ with the Schrödinger wave theory.

The last chapter brings a five-dimensional interpretation. A bibliography is found at the end.

The author has given an exposition of the relations between his theory and wave mechanics in more recent papers on the *Geometry of two-component spinors* and the *Geometry of four-component spinors* (Proceedings of the National Academy of Sciences, April 19, 1933, pp. 462–474, 503–517). These papers, together with the book on projective relativity, present a complete unified field theory.

Veblen's interpretation of projective relativity has many points in common with and also certain differences from the similar theory presented by Schouten and Van Dantzig in several recent papers published in the Zeitschrift für Physik (see also Annals of Mathematics, vol. 34 (1933), p. 271). One of the chief points of difference between the two theories, which both interpret gravitation, electromagnetism, and wave mechanics, lies in the fact that the latter authors use a projective connection with a set of five equivalent homogeneous coordinates, while in Veblen's interpretation one coordinate, x^0, is singled out from the beginning. Both theories can be made to embrace previous attempts to construct a "curved" projective geometry and a projective relativity.

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