NOTE ON A PROBLEM OF FRÉCHET*

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1. *Introduction.* Fréchet has proposed the following problem: Characterize the most general space in which there exists a non-constant real-valued continuous function. Its solution has been given by Urysohn† for the spaces of Hausdorff and by Chittenden‡ for topological spaces.

However, in his generalization, Chittenden used for his definition of a continuous function a neighborhood definition§ of Fréchet which is not entirely adequate for the general topological space. That the definition is not adequate is easily shown by the following example. Let the space (P, K) be a set of points corresponding to the open interval $[0, 1]$. For any set E of (P, K), $K(E)=R(E)+J(E)$. A point x is in $R(E)$ if x is in a derived set of E under the metric relationship. A point x is in $J(E)$ if E contains both x and the open interval $[0, x/2]$. A neighborhood of x is any set to which x is interior. The correspondence $f(x)=x$ between the space (P, K) and the space (P, R) determines under the Fréchet definition a biunivocal bicontinuous transformation, although the two spaces are not homeomorphic.

The purpose of this paper is to solve the problem for the general topological space using the Sierpinski definition|| of a continuous transformation. The notation used will be that of Chittenden.¶

2. *Thoroughly Interior.* A point a is said to be *thoroughly in-*

* Presented to the Society, July 18, 1933.

† Urysohn, *Über de Mächtigkeit der zusammenhängenden Mengen*, Mathematische Annalen, vol. 94 (1925), p. 290.

‡ Chittenden, *On general topology and the relation of the properties of the class of all continuous functions to the properties of space*, Transactions of this Society, vol. 31 (1929), p. 310.

§ Chittenden, loc. cit., p. 309.

|| Sierpinski, *La notion de dérivée comme base d'une théorie des ensembles abstraits*, Mathematische Annalen, vol. 97 (1926), pp. 321–337.

¶ Chittenden, loc. cit. Also see Stephens, *Continuous transformations of abstract spaces*, Transactions of this Society, vol. 34 (1932), p. 395.

terior to a set E if a is contained in E and not contained in $L[C(E)+a]$.* A set A is thoroughly interior to a set E if each point of A is thoroughly interior to E. A set E is thoroughly open if each point of E is thoroughly interior to E. The complement of a thoroughly open set we shall call a thoroughly closed set. It is easily seen that in a neighborhood space every open (closed) set is thoroughly open (closed). Furthermore the sum of an infinite number of thoroughly open (closed) sets is thoroughly open (closed).

THEOREM 1. *If f is a continuous transformation such that $f(P) = Q$, and b is a point thoroughly interior to $B \subset Q$, then $g(b)$ is thoroughly interior to $g(B)$.*†

The proof is by contradiction. Assume $g(b)$ is not thoroughly interior to $g(B)$. Then there exists a point a of $g(b)$ and a set $E \subset C[g(B)]$ such that $a \subset (E+a)'$ but not in E' since $g(b)$ is interior to $g(B)$.‡ Then $f(a) \subset f(E) + [f(E+a)]' \subset [f(E-a)]$, which is impossible.

As a corollary we have the following theorem.

THEOREM 2. *A necessary condition that a transformation be continuous is that the inverse of every thoroughly open (closed) set be thoroughly open (closed).*

THEOREM 3. *If a space (P, K) has no singular points,§ then there exists a V-space (P, W) which is a biunivocal continuous transformation of P in such a way that thoroughly open (closed) sets are transformed into open (completely closed) sets.*

The space (P, W) is defined on the set P as follows. For each point a of P, if $a \subset K(E)$ in (P, K), then $a \subset W(B)$ in (P, W), where B is any set containing $(E-a)$. The space (P, W) is a neighborhood space which is the biunivocal continuous transform of (P, K) under the transformation f which carries each point into itself.|| Furthermore, if any set O is thoroughly K-

* $L(E)$ is the set of all K-points of all subsets of E. Chittenden, loc. cit., p. 294. $C(E)$ is the complement of E in P.

† By $g(b)$ is denoted the set of all points of P to which b of Q corresponds under f.

‡ Stephens, loc. cit., p. 397.

§ A point is a singular point if $a \subset K(a)$. Stephens, loc. cit., p. 396.

|| Stephens, loc. cit., p. 405.

open, it is thoroughly W-open. For suppose O is thoroughly K-open but not thoroughly W-open. Then there exists a point a of O such that $a \subset W(B)$ where $B \subset C(O)$. Hence $a \subset K(E)$ where $(E-a) \subset B \subset C(O)$, and O cannot be thoroughly K-open, contrary to hypothesis. The proof for thoroughly closed sets is obvious.

3. *Normal Families of Thoroughly Open Sets.* A family of thoroughly open sets G is normal† provided

(a) there exist two sets G_0, G_1 of the family which are distinct and such that $G_0 \subset G_1$;

(b) if G_0, G_1 are any two sets of the family such that $G_0^* \subset G_1$,‡ there is a set G such that

$$G_0^* \subset G \subset G^* \subset G_1.$$

THEOREM 4. *If the continuous transform Q of a space P contains a normal family of thoroughly open sets, then the original space P contains such a family.*

Since $G_0 \subset G_1$ are thoroughly open sets, $g(G_0) \subset g(G_1)$ are thoroughly open sets satisfying condition (a). For the inverse sets of

$$G_0^* \subset G \subset G^* \subset G_1$$

we have the relation

$$g(G_0^*) \subset g(G) \subset g(G^*) \subset g(G_1),$$

where $g(G_0^*)$ and $g(G^*)$ are thoroughly closed sets. We have then§

$$g(G_0) \subset [g(G_0)]^* \subset g(G_0^*) \subset g(G) \subset [g(G)]^* \subset g(G^*) \subset g(G_1).$$

Hence P contains a normal family of thoroughly open sets.

THEOREM 5. *If a space P contains no singular points and possesses a normal family of thoroughly open sets, there exists a V-space Q having a normal family of open sets, which is a biunivocal continuous transform of P.*

By Theorem 3, there is a space which is a biunivocal continuous transform Q of P such that thoroughly open (closed) sets

† Chittenden, loc. cit., p. 310.

‡ The set G_0^* is the least thoroughly closed set containing G_0.

§ $g(G_0)^* \subset g(G_0^*)$ since the product of a finite or infinite family of thoroughly closed sets is thoroughly closed.

are transformed into open (completely closed) sets. It remains to show that Q contains a normal family of open sets.

Let G be the normal family of thoroughly open sets in P. Then $f(G_0)$ and $f(G_1)$ satisfy condition (a). Also $f(G_0) \subset f(G_0^*) \subset f(G) \subset f(G^*) \subset f(G_1)$, from which we have

$$f(G_0) \subset [f(G_0)]^* \subset f(G) \subset [f(G)]^* \subset f(G_1).$$

Hence Q possesses a normal family of open sets and the theorem is proved.

4. *Non-Constant Continuous Functions.* We are now prepared to prove the following generalization of the Urysohn theorem.

THEOREM 6. *A necessary and sufficient condition that a topological space admit the existence of a non-constant real-valued continuous function is that it contain a normal family of thoroughly open sets and that it contain no singular points.*

The necessity of the normality requirement is immediate from Theorem 4. I† have proved that the space P must contain no singular point.

To prove that the conditions are sufficient we use Theorem 5 to obtain a V-space possessing a normal family of open sets. But now the Fréchet definition holds and the theorem of Chittenden‡ is valid.

It should be noted that the requirement of normality is significant only in connected spaces. The Urysohn corollary to this theorem as stated by Chittenden§ holds.

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† Stephens, loc. cit., p. 405.
‡ Chittenden, loc. cit., p. 310.
§ Chittenden, loc. cit., p. 310.