parison of the limits given by (2) and by Tchebycheff's theorem, Fig. 2 has been prepared. This shows the values of $1-P_{t\sigma}$ plotted against t, the several curves corresponding to various values of ρ as indicated. The dotted line gives the limits from Tchebycheff's theorem.

BELL TELEPHONE LABORATORIES

CHARACTERISTICS OF MULTIPLE CURVES AND THEIR RESIDUALS*

BY T. R. HOLLCROFT

Salmon† obtained formulas relating the characteristics of two curves which together form the complete intersection of two algebraic surfaces when one of the curves is double on one of the surfaces. In this paper, by a generalization of Salmon's method, the relations between the characteristics of two such curves are found when one of the curves is of given multiplicity on each of the two surfaces. Such a formula is useful in studying a system of surfaces with a multiple basis curve. It was this need for it that led to its derivation.

Consider two algebraic surfaces f_1 and f_2 of orders μ_1 and μ_2, respectively, whose complete intersection consists of two curves C_1, C_2 of orders n_1, n_2; ranks r_1, r_2; genera p_1, p_2; and with h_1, h_2 apparent double points, respectively. Assume that C_1 is of multiplicity i_1 on f_1 and i_2 on f_2 and also that C_1 itself is the complete intersection curve of two surfaces. C_1 (counted simply) and C_2 have t actual intersections and n_1n_2-t apparent intersections.

Consider a third surface f_3 of order μ_3 passing simply through C_1 but not through C_2. The equivalence E of C_1 on the three surfaces f_1, f_2, f_3 is‡

$$E = n_1(i_2\mu_1 + i_1\mu_2 + i_1i_2\mu_3 - 2i_1i_2) - i_1i_2r_1.$$

---

*Presented to the Society, April 14, 1933.

† Salmon, *Geometry of Three Dimensions*, 4th ed., 1882, p. 322.

‡ M. Noether, *Sulle curve multiple di superficie algebriche*, Annali di Matematica, (2), vol. 5 (1871), p. 166.

The number of points of intersection of f_1, f_2, f_3 outside of C_1 is, therefore, $\mu_1\mu_2\mu_3-E$.

The surfaces f_1 and f_2 intersect in the curve C_1 counted i_1i_2 times and the residual curve C_2 of order $n_2=\mu_1\mu_2-i_1i_2n_1$. Since f_3 does not pass through C_2, the number of intersections of f_1, f_2, f_3 outside of C_1 equals the number of intersections of C_2 and f_3 less the number t of actual intersections of C_1 and C_2. Equating the two expressions for the number of intersections of f_1, f_2, f_3 outside of C_1, we have

$$\mu_3(\mu_1\mu_2 - i_1i_2n_1) - t = \mu_1\mu_2\mu_3 - E,$$

and solving for t, we find

$$t = n_1(i_2\mu_1 + i_1\mu_2 - 2i_1i_2) - i_1i_2r_1. \tag{A}$$

In order to find an expression for t in terms of the characteristics of C_2, consider the locus F of points whose polar planes with respect to f_1 and f_2 meet an arbitrary line at one point. This surface F is of order $\mu_1+\mu_2-2$.* Since the curve C_1 is i_1-fold on f_1 and i_2-fold on f_2, it is of multiplicity i_1+i_2-2 on F.

The residual curve C_2 meets F in $n_2(\mu_1+\mu_2-2)$ points. Since F contains C_1 as an (i_1+i_2-2)-fold curve and since the surfaces f_1 and f_2 have contact at each of the t actual intersections of C_1 and C_2, these intersections count as $(i_1+i_2-1)t$ intersections of C_2 and F. Also, C_2 and F intersect in the r_2 points of contact of the r_2 tangents to C_2 which meet the arbitrary line associated with F. Equating the two expressions for the number of intersections of C_2 and F, we find

$$(i_1 + i_2 - 1)t + r_2 = n_2(\mu_1 + \mu_2 - 2). \tag{B}$$

Eliminating t from (A) and (B) and simplifying, we obtain

$$\begin{aligned} r_2 - i_1i_2(i_1 + i_2 - 1)r_1 &= n_2(\mu_1 + \mu_2 - 2) \\ &\quad - n_1(i_1 + i_2 - 1)(i_2\mu_1 + i_1\mu_2 - 2i_1i_2). \end{aligned} \tag{1}$$

Substituting $r_k=n_k(n_k-1)-2h_k$, $(k=1, 2)$, in (1), we find the following relation involving apparent double points:

$$\begin{aligned} 2h_2 - 2i_1i_2(i_1 + i_2 - 1)h_1 &= n_1(i_1 + i_2 - 1)\left[i_2\mu_1 + i_1\mu_2\right. \\ &\quad \left. - i_1i_2(n_1 + 1)\right] - n_2(\mu_1 + \mu_2 - n_2 - 1). \end{aligned} \tag{2}$$

* Salmon, loc. cit., pp. 308–309.

Substituting $2h_k=(n_k-1)(n_k-2)-2p_k$, $(k=1, 2)$, in (2), we obtain the following relation involving the genus:

$$
(3)\quad \begin{aligned} 2(p_2 - 1) - 2i_1i_2(i_1 + i_2 - 1)(p_1 - 1) &= n_2(\mu_1 + \mu_2 - 4) \\ &\quad - n_1(i_1 + i_2 - 1)(i_2\mu_1 + i_1\mu_2 - 4i_1i_2). \end{aligned}
$$

Formulas (1), (2), (3) define the characteristics of either curve when those of the other and the multiplicities of C_1 are known. The order and multiplicities of C_1 must satisfy the inequality

$$
n_1i_k(i_k - 1) \leqq (\mu_k - 1)(\mu_k - 2), \qquad (k = 1, 2).*
$$

When the multiple curve C_1 is the complete intersection of f_1 and f_2, the equation $\mu_1\mu_2=i_1i_2n$ and the above formulas become (on setting $n_2=r_2=0$ and dropping the subscripts of the characteristics of C_1):

$$
(1') \qquad i_1i_2r = n(i_2\mu_1 + i_1\mu_2 - 2i_1i_2),
$$

$$
(2') \qquad 2i_1i_2h = n(\mu_1 - i_1)(\mu_2 - i_2),
$$

$$
(3') \qquad 2i_1i_2(p - 1) = n(i_2\mu_1 + i_1\mu_2 - 4i_1i_2).
$$

The above methods and formulas are valid for all values of order and multiplicity satisfying the necessary conditions, provided that C_1 counted i_1i_2 times and C_2 together constitute the complete intersection of f_1 and f_2 and provided further that C_1 is itself a complete intersection of two surfaces.

The assumption that C_1 is the complete intersection of two surfaces is necessary because the equivalence formula used for C_1 was derived by Noether† only for a complete intersection curve. However, it has been stated by Hudson,‡ on evidence not sufficient for proof, that this equivalence formula is very probably valid for all space curves of given order and rank whether complete intersections or not. It is, therefore, probable that the formulas derived above hold for all space curves, but no proof of this is available.

WELLS COLLEGE

* For $i_k>2$ and large values of μ_k, this inequality must be replaced by another. See T. R. Hollcroft, *Multiple points of algebraic curves*, this Bulletin, vol. 35 (1929), pp. 848–849.

† M. Noether, loc. cit.

‡ H. Hudson, *Cremona Transformations*, 1927, p. 221, end of §15.