ON A COVARIANT DIFFERENTIATION PROCESS: PAPER II*

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1. *Introduction*. It is the purpose of this note to construct analogs of the parameters gradient, divergence, and curl, and to establish a few of their more salient properties.

2. *Notation*. In addition to the notation used in I,† we shall employ the symbols $|$, θ to indicate ordinary covariant and Synge-Taylor differentiation, respectively.

3. *The Invariants*. Evidently, if $S(x, x', \cdots, x^{(m)})$ is a scalar, then the quantities $S_{,\alpha}$ are the components of a vector. Likewise, if $V^\alpha(x, x', \cdots, x^{(m)})$, V_α $(V_\alpha = f_{\alpha\beta}V^\beta)$ are the contravariant and covariant descriptions of a vector, and $A_{\alpha\beta}$ a second order tensor, then $V^\alpha{}_{,\alpha}$ is a scalar, $V_{\alpha,\beta} - V_{\beta,\alpha}$ a skew symmetric tensor, and $A_{\alpha\beta,\gamma} + A_{\gamma\alpha,\beta} + A_{\beta\gamma,\alpha}$ a tensor of the third order. Furthermore, if n, the dimensionality of the space, is three, and $\epsilon^{\alpha\beta\gamma}$ represents the product of $|f_{\alpha\beta}|^{-1/2}$ and the corresponding component of the contravariant e system, then $\epsilon^{\alpha\beta\gamma}\, V_{\beta,\gamma}$ is a vector. The symbols $e^{\alpha\beta\gamma}$ are skew symmetric in each pair of indices and e^{123} is unity.

A certain regularity appears if $m > 2$ or if the affine connection is that of Riemannian geometry, for example, $V_{\alpha,\beta} = f_{\alpha\gamma}V^\gamma{}_{,\beta}$, and whenever either of these cases prevails we shall employ a special symbolism. Specifically, GS shall represent the vector $S_{,\alpha}$, DV the scalar $V^\alpha{}_{,\alpha}$, and if n is three, CV the vector $\epsilon^{\alpha\beta\gamma}\, V_{\beta,\gamma}$.

In virtue of these definitions and the formal equivalence of certain of the rules of operation of our process and those of partial differentiation, we may take over many of the identities of vector analysis; for example,

$$CGS \equiv 0; \qquad DCV \equiv 0; \qquad G(S + s) \equiv GS + Gs;$$

$$D(SV) \equiv (GS)\cdot V + SDV; C(SV) \equiv (GS) \times V + SCV; \text{ etc.}$$

The first of these relationships suggests the following theorem.

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* Presented to the Society, March 25, 1932.

† The preceding note, this Bulletin, vol. 37 (1931), p. 731.

THEOREM. *A necessary and sufficient condition that a function s exist such that* $S_{,\alpha} = V_\alpha$, *is* $V_{\alpha,\beta} - V_{\beta,\alpha} \equiv 0$.

In proving the sufficiency of this condition (the necessity is obvious) we shall simplify the writing by restricting ourselves to the case $m = 2$. The cases $m = 3$, $m > 3$ are somewhat simpler and may be treated as in the following demonstration.

PROOF. Let us replace the given equations with the equivalent set of n homogeneous partial differential equations,

$$A_\alpha \Phi = 0, \quad \left(A_\alpha \equiv \frac{\partial}{\partial x'^\alpha} - 2 \begin{Bmatrix} \sigma \\ \alpha \end{Bmatrix} \frac{\partial}{\partial x''^\sigma} + V_\alpha \frac{\partial}{\partial s} \right),$$

in the dependent variable Φ and the $2n+1$ independent variables x', x'', s. Obviously, these equations are independent. If, in addition, they are Jacobian complete, that is, if the alternants

$$(A_\alpha A_\beta)\Phi \equiv - \left(A_\alpha 2 \begin{Bmatrix} \Lambda \\ \beta \end{Bmatrix} \right) \frac{\partial \Phi}{\partial x''^\Lambda} + V_{\beta,\alpha} \frac{\partial \Phi}{\partial s}$$
$$+ \left(A_\beta 2 \begin{Bmatrix} \Lambda \\ \alpha \end{Bmatrix} \right) \frac{\partial \Phi}{\partial x''^\Lambda} - V_{\alpha,\beta} \frac{\partial \Phi}{\partial s}$$

vanish, then we may conclude that the set $A_\alpha \Phi = 0$ has a solution,* such that $\Phi = 0$ may be solved for s. The function s so obtained is the required scalar.

As a matter of fact, a demonstration that the sum $-A_\alpha 2 \left\{ {\Lambda \atop \beta} \right\} + A_\beta 2 \left\{ {\Lambda \atop \alpha} \right\}$ is zero is a part of the proof of the commutative property of the differentiation process,† and so $(A_\alpha A_\beta)\, \Phi$ reduces to $(V_{\beta,\alpha} - V_{\alpha,\beta})\, \partial \Phi / \partial s$, which vanishes by hypothesis.

A second theorem,‡ which indicates a similarity between divergence and $PA_{\alpha\beta\gamma}$, $(PA_{\alpha\beta\gamma} \equiv A_{\alpha\beta,\gamma} + A_{\gamma\alpha,\beta} + A_{\beta\gamma,\alpha})$, and between curl and $V_{\alpha,\beta} - V_{\beta,\alpha}$ may be stated as follows.

THEOREM. *If* $A_{\alpha\beta}$ *is a skew-symmetric tensor, then the necessary and sufficient condition that a covariant vector* Φ_α *exist, such that*

* See Goursat, *Mathematical Analysis*, vol. 2, translated by Hedrick and Dunkel, part 2, pp. 265–270; A. Cohen, *The Lie Theory of One Parameter Groups*, pp. 109–111.

† See I, p. 733.

‡ See L. P. Eisenhart, *Condition that a tensor be the curl of a vector*, this Bulletin, vol. 28 (1922), p. 425.

$$A_{\alpha\beta} = \Phi_{\alpha,\beta} - \Phi_{\beta,\alpha}, \tag{1}$$

is that

$$PA_{\alpha\beta\gamma} = 0. \tag{2}$$

PROOF. Let Φ_1 and Φ_2 be two functions which satisfy the equation $A_{12} = \Phi_{1,2} - \Phi_{2,1}$. In virtue of (2) there is a function $\Psi(x, x', x'', s)$, such that the equality

$$\frac{\partial\Psi}{\partial x'^{\alpha}} - 2\left\{\begin{matrix}\sigma\\ \alpha\end{matrix}\right\}\frac{\partial\Psi}{\partial x''^{\sigma}} + (\Phi_{\alpha,3} - A_{\alpha 3})\frac{\partial\Psi}{\partial s} = 0, \qquad (\alpha = 1, 2),$$

holds, for the alternant in question reduces to PA_{123}. Moreover, $\Psi(x, x', x'', s) = 0$, $s = \Phi_3$ define a function Φ_3, which satisfies the required relations $A_{13} = \Phi_{1,3} - \Phi_{3,1}$; $A_{23} = \Phi_{2,3} - \Phi_{3,2}$. Evidently Φ_4, and other successive Φ functions, may be found similarly.

Finally, we note that if we transform Φ_α as a covariant vector, then both (1) and (2) will be tensor equations and consequently valid in all coordinate systems.

An additional characteristic of the invariant GS, somewhat analogous to a certain property of the gradient, namely, that the critical points of $z(x, y)$ are determined by grad $z = 0$, is expressed by the following theorem.

THEOREM. *The equation $Gf^{(m)}(x, x') = 0$ determines the extremal curves associated with $\int f dt$.*

PROOF. By differentiating $f(x, x')$ repeatedly with respect to the parameter and representing each time with R those terms which do not contribute to the corresponding $Gf^{(m)}$, we have

$$\begin{aligned} f' &= x'^{\alpha} f_{x^{\alpha}} + x''^{\alpha} f_{\alpha}; \\ f'' &= x''^{\alpha}(f_{x^{\alpha}} + 2x'^{\beta} f_{\alpha x^{\beta}} + x''^{\beta} f_{\alpha\beta}) + x'''^{\alpha} f_{\alpha} + R; \\ f^{(m)} &= x^{(m)\alpha}\left[f_{x^{\alpha}} + m(f_{\alpha})'\right] + x^{(m+1)\alpha} f_{\alpha} + R, \qquad (m > 2). \end{aligned}$$

Applying our process to the first of these and expanding, we obtain

$$f'_{,\gamma} = f_{x^{\gamma}} + x'^{\alpha} f_{x^{\alpha}\gamma} + x''^{\alpha} f_{\alpha\gamma} - 2 f_{\sigma} f^{\rho\sigma}\left\{x'^{\delta}[\gamma\delta, \rho] + \tfrac{1}{2} x''^{\tau} f_{\gamma\rho\tau}\right\}.$$

But the followings relations hold:

$$f_{\sigma} f^{\rho\tau} = x'^{\alpha} f_{\sigma\alpha} f^{\rho\sigma};\ x'^{\rho} f_{\gamma\rho\tau} = 0;\ x'^{\delta} x'^{\rho}\left[f_{\rho\delta x^{\gamma}} + f_{\gamma\rho x^{\delta}} - f_{\gamma\delta x^{\rho}}\right] = 2 f_{x^{\gamma}};$$

and therefore $f'_{,\gamma}$ is the Euler tensor, $-f_{x^{\gamma}} + (f_{\gamma})'$. Similarly, we find that $f^{(m)}{}_{,\gamma} = m\left[-f_{x^{\gamma}} + (f_{\gamma})'\right]$.

A second application of the process reveals the fact that the covariant Euler tensor E_α is a constant with respect to our derivative, thus:

$$E_\alpha = x''^\gamma f_{\alpha\gamma} + x'^\gamma x'^\delta[\gamma\delta, \alpha],$$

$$E_{\alpha,\beta} = x''^\gamma f_{\alpha\gamma\beta} + 2x'^\gamma[\gamma\beta, \alpha] - 2f_{\alpha\sigma}f^{\sigma\rho}(x'^\gamma[\gamma\beta, \rho] + \tfrac{1}{2}x''^\gamma f_{\rho\gamma\beta}) \equiv 0,$$

while

$$E^\alpha{}_{,\beta} = (f^{\alpha\gamma}E_\gamma)_{,\beta} = f^{\alpha\gamma}{}_{,\beta}\, E_\gamma = f^{\alpha\gamma}_{x'^\beta}E_\gamma = - f^{\alpha\gamma}f_{\gamma\delta\beta}E^\delta.$$

Incidentally, the tensor $f^{\alpha\gamma}{}_{x'^\beta}E_\gamma$ appears in $(\theta E_\alpha)_{,\beta}$ also, for

$$\begin{aligned}\theta E_\alpha = {} & (x''^\gamma f_{\alpha\gamma})' + 2x''^\gamma x'^\delta[\gamma\delta, \alpha] + R \\ & - \left\{x''^\gamma f_{\sigma\gamma} + x'^\gamma x'^\delta[\gamma\delta, \sigma]\right\}\left\{x'^\mu[\mu\alpha, \rho] + \tfrac{1}{2}x''^\tau f_{\alpha\tau\rho}\right\}f^{\rho\sigma};\end{aligned}$$

and consequently

$$\begin{aligned}(\theta E_\alpha)_{,\beta} &= f_{\alpha\beta}' + x''^\gamma f_{\alpha\gamma\beta} + 2x'^\delta[\beta\delta, \alpha] - \left\{x'^\mu[\alpha\mu, \beta] + \tfrac{1}{2}x''^\tau f_{\alpha\tau\beta}\right\} \\ &\quad - \tfrac{1}{2}E^\rho f_{\alpha\beta\rho} - 3f_{\alpha\sigma}f^{\rho\sigma}\left\{x'^\mu[\beta\mu, \rho] + \tfrac{1}{2}x''^\tau f_{\beta\tau\rho}\right\} \\ &= -\tfrac{1}{2}E^\rho f_{\alpha\beta\rho} = \tfrac{1}{2}f^{\rho\sigma}_{x'^\beta}E_\sigma f_{\alpha\rho};\end{aligned}$$

also

$$(\theta E^\alpha)_{,\beta} = [\theta(f^{\alpha\gamma}E_\gamma)]_{,\beta} = f^{\alpha\gamma}(\theta E_\gamma)_{,\beta} = -\tfrac{1}{2}E^\rho f_{\gamma\beta\rho}f^{\alpha\gamma} = \tfrac{1}{2}E_\gamma f^{\alpha\gamma}{}_{x'^\beta}.$$

Obviously whenever the components of a tensor do not involve x'', we may make our derivative applicable by first applying the θ process. To illustrate, suppose that we have given S_{x^α}, the gradient of a scalar point function, and let us confine ourselves to Riemannian geometry—the θ process in this case reduces to Levi-Civita's derivative. Thus, by differentiating and employing the relationships $\theta x'^\gamma = E^\gamma$; $E^\gamma{}_{,\beta} = 0$, we find

$$\theta S_{x^\alpha} = x'^\gamma S_{x^\alpha|\gamma};\ \theta^2 S_{x^\alpha} = (\theta x'^\gamma)S_{x^\alpha|\gamma} + x'^\gamma x'^\delta S_{x^\alpha|\gamma|\delta};$$

$$(\theta^2 S_{x^\alpha})_{,\beta} = 2x'^\gamma S_{x^\alpha|\gamma|\beta} = 2x'^\gamma S_{x^\gamma|\alpha|\beta}.$$

Finally, it is interesting to note that the *curl* of $\theta^2 S_{x^\alpha}$ involves the Riemann-Christoffel tensor, thus

$$(\theta^2 S_{x^\alpha})_{,\beta} - (\theta^2 S_{x^\beta})_{,\alpha} = 2x'^\gamma S_{x^\delta}\overset{\delta}{R}_{\gamma\alpha\beta}.$$

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