AN INTEGRAL EQUATION WITH SYMMETRIC KERNELS*

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It is the purpose of this note to investigate conditions necessary and sufficient for the solution of the integral equation

$$(1) \quad \int_a^b A(x, s)X(s, y)ds + \int_a^b X(x, s)B(s, y)ds = C(x, y),$$

where the kernels $A(x, y)$ and $B(x, y)$ are considered to be symmetric. We further restrict our functions of two variables to be continuous throughout the fundamental interval (a, b).

An equation of the type (1) will not in general admit a solution. However, under certain quite restrictive conditions on the function $C(x, y)$, a solution may be obtained. To determine these conditions, we may readily verify from the classical theory of integral equations that every function $C(x, y)$, for which a function $X(x, y)$ exists such that (1) is true, is developable in a uniformly convergent series

$$(2) \qquad C(x, y) = \sum_{i=1}^{\infty} \left\{ \frac{\alpha_i(x)\bar{\alpha}_i(y)}{\alpha_i} + \frac{\bar{\beta}_i(x)\beta_i(y)}{\beta_i} \right\},$$

where

$$(3) \quad \bar{\alpha}_i(y) = \int_a^b \alpha_i(s)X(s, y)ds, \quad \bar{\beta}_i(x) = \int_a^b X(x, s)\beta_i(s)ds,$$

and where $\{\alpha_i, \alpha_i(s)\}$ and $\{\beta_i, \beta_i(s)\}$ are the characteristic values and characteristic functions of the kernels $A(x, y)$ and $B(x, y)$, respectively. To justify this conclusion, it is sufficient to note that the series for the iterated kernel

$$A^{(2)}(x, y) = \sum_{i=1}^{\infty} \frac{\alpha_i(x)\alpha_i(y)}{\alpha_i^2}, \qquad A^{(2)}(x, x) = \sum_{i=1}^{\infty} \frac{\alpha_i^2(x)}{\alpha_i^2},$$

converge uniformly and absolutely, which, in view of the boundedness of $\sum_{i=1}^{\infty}\bar{\alpha}_i^2(y)$, implies the uniform and absolute

* Presented to the Society, March 18, 1933.

convergence of $\sum_{i=1}^{\infty}\alpha_i(x)\bar{\alpha}_i(y)/\alpha_i$. A similar argument applies for the other terms of (2).

This observation shows us that a *necessary* condition for a solution of the equation (1) under our hypotheses is that $C(x, y)$ be expressible linearly in terms of the characteristic functions of the kernels $A(x, y)$ and $B(x, y)$.

In the following theorems we shall have occasion to refer to a characteristic root of (1). *If α_i and β_i are the respective characteristic values of the symmetric kernels $A(x, y)$ and $B(x, y)$ and if $\alpha_i = -\beta_k$ (any i and any k), we say that α_i (or $-\beta_k$) is a characteristic root with respect to these kernels.*

We may then state the following theorem.

THEOREM 1. *Assuming that* (1) *has no characteristic roots and that $C(x, y)$ has the necessary form*

$$C(x, y) = \sum_{i=1}^{\infty}\{\alpha_i(x)A_i(y) + B_i(x)\beta_i(y)\}, \tag{4}$$

then if the series

$$\sum_{i=1}^{\infty}\{\alpha_i\alpha_i(x)\overline{A}_i(y) + \beta_i\overline{B}_i(x)\beta_i(y)\} \tag{5}$$

is uniformly convergent, where $\overline{A}_i(y)$ and $\overline{B}_i(x)$ are defined by

$$\begin{cases} A_i(y) = \overline{A}_i(y) + \alpha_i\int_a^b \overline{A}_i(s)B(s, y)ds, \\ B_i(x) = \overline{B}_i(x) + \beta_i\int_a^b A(x, s)\overline{B}_i(s)ds, \end{cases} \tag{6}$$

it is a solution of (1).

This theorem is established directly by the substitution of (5) into (1) and noting by our hypothesis on the characteristic roots that the functions $\overline{A}_i(y)$ and $\overline{B}_i(x)$ are determined uniquely by (6).

It is evident that a more general solution of (1) may be obtained by adding to (5) any non-vanishing solutions of the equation

$$\int_a^b A(x, s)Z(s, y)ds + \int_a^b Z(x, s)B(s, y)ds = 0. \tag{7}$$

A treatment of the non-vanishing solutions of equations essentially of the form (7) has been made by Lauricella.* We wish here only to point out the following result.

THEOREM 2. *If* (7) *has no characteristic roots, then all non-vanishing solutions of* (7) *have the property*

$$(8)\qquad \begin{cases} \int_a^b \alpha_i(s)Z(s, y)ds = 0, & (i = 1, 2, \cdots, n), \\ \int_a^b Z(x, s)\beta_i(s)ds = 0, & (i = 1, 2, \cdots, n). \end{cases}$$

We may verify this theorem at once on multiplying (7) by $\alpha_i(x)$, integrating with respect to x, and applying

$$(9)\qquad \alpha_i(x) = \alpha_i \int_a^b A(x, s)\alpha_i(s)ds.$$

This gives us

$$\left\{\int_a^b \alpha_i(s)Z(s, y)ds\right\} + \alpha_i \int_a^b \left\{\int_a^b \alpha_i(s)Z(s, t)ds\right\} B(t, y)dt = 0;$$

and hence

$$\int_a^b \alpha_i(s)Z(s, y)ds = 0.$$

A similar procedure establishes the second equation of (8).

Let us now consider the case in which there exist characteristic roots of (1). On multiplying (1) by $\alpha_i(x)\beta_k(y)$, (where $\alpha_i = -\beta_k$), and integrating on x and y, we obtain further *necessary* conditions on the coefficients of $C(x, y)$, namely,

$$(10)\qquad \int_a^b A_i(s)\beta_k(s)ds + \int_a^b B_k(s)\alpha_i(s)ds = 0.$$

With respect to these conditions, let us assume for the moment that the coefficients $B_k(s)$ are assigned arbitrarily and that the

* See Lauricella, *Sopra le funzioni permutabili di* 2[a] *specie*, Lincei Rendconti, 1[e] Sem., (1913).

$A_i(s)$ are determined by (10). This means that the $A_i(s)$ will have the form

$$
\begin{aligned}
A_i(s) &= \textstyle\sum_k a_{ik}\beta_k(s) + A_i'(s),\\
a_{ik} &= -\int_a^b \alpha_i(t)B_k(t)dt,
\end{aligned}
\tag{11}
$$

where the summation is taken over all k for which $\alpha_i = -\beta_k$, and where $A_i'(s)$ is an arbitrary function which is orthogonal to all the corresponding $\beta_k(s)$. With this change we may rewrite the form of $C(x, y)$ as

$$
\begin{aligned}
C(x, y) &= \sum_{i=1}^{\infty}\{\alpha_i(x)[\textstyle\sum_k a_{ik}\beta_k(y) + A_i'(y)] + B_i(x)\beta_i(y)\}\\
&= \sum_{i=1}^{\infty}\{\alpha_i(x)A_i'(y) + B_i'(x)\beta_i(y)\}.
\end{aligned}
\tag{12}
$$

Now considering the conditions (10) for the coefficients of this new form of $C(x, y)$, we have at once, since $\int_a^b A_i'(s)\beta_k(s)ds = 0$, the additional relation

$$
\int_a^b B_k'(s)\alpha_i(s)ds = 0.
$$

For this reason we may assume, without loss of generality, that *a necessary condition for the existence of a solution to equation* (1) *is that the function* $C(x, y)$ *have the form* (4), *where*

$$
\int_a^b A_i(s)\beta_k(s)ds = 0, \qquad \int_a^b \alpha_i(s)B_k(s)ds = 0
\tag{13}
$$

for all i *and* k *such that* $\alpha_i = -\beta_k$. We may then generalize Theorem 1 to read as follows.

THEOREM 1′. *Let the function* $C(x, y)$ *have the necessary form* (4) *satisfying the additional conditions* (13); *then if the series* (5) *is uniformly convergent, it is a solution of* (1).

The only point in question with respect to the refinements made in the above theorem is whether the equations (6) will always permit a solution. We observe, however, that for $\alpha_i = -\beta_k$ the necessary and sufficient conditions for the solution of (6) are precisely the conditions (13).

Considering the equation (7) with the introduction of characteristic roots, we have a somewhat different situation. Consider, for example, that $\alpha_i = -\beta_k$. Multiplying (7) by α_i, we have

$$\alpha_i \int_a^b A(x, s)Z(s, y)ds = \beta_k \int_a^b Z(x, s)B(s, y)ds. \tag{14}$$

Multiplying (14) by $\alpha_i(x)$ and $\beta_k(y)$ separately and integrating, we have by (9)

$$\begin{cases} \int_a^b \alpha_i(s)Z(s, y)ds = \beta_k \int_a^b \int_a^b \alpha_i(s)Z(s, t)B(t, y)dsdt, \\ \int_a^b Z(x, t)\beta_k(t)dt = \alpha_i \int_a^b \int_a^b A(x, s)Z(s, t)\beta_k(t)dsdt. \end{cases} \tag{15}$$

These equations imply that

$$\int_a^b \alpha_i(s)Z(s, y)ds = \sum_k c_k\beta_k(y),$$

$$\int_a^b Z(x, t)\beta_k(t)dt = \sum_i c_i' \alpha_i(x),$$

where the summations extend over all k and i respectively for which $\alpha_i = -\beta_k$. These conditions on $Z(x, y)$ lead us to the following generalization of Theorem 2.

THEOREM 2'. *All non-vanishing solutions of* (7) *are of the form*

$$\sum_{i,k} c_{ik}\alpha_i(x)\beta_k(y) + Z(x, y), \qquad (c_{ik} = \text{const.}), \tag{16}$$

where i and k range over all indices having the property $\alpha_i = -\beta_k$, and where $Z(x, y)$ satisfies (7).

Clearly the solution, (5), of (1) is made more general by the addition of terms (16) and the sufficiency as a solution is obvious.

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