ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

266. Professor T. R. Hollcroft: *The web of algebraic surfaces with basis points.*

The web of algebraic surfaces with α basis points P_i is defined by the equation $\Sigma\lambda_k f_k=0$, $(k=1, 2, 3, 4)$, in which the f_k are surfaces, each with a multiple point of order r_i at each of the given points P_i, $(i=1, \cdots, \alpha)$, respectively. By means of the (1,1) correspondence existing between the planes of space and the surfaces of this web, an involution of order $n^3-\Sigma r_i^3$ is established. The characteristics of the branch-point and coincidence surfaces of this involution are obtained, and, from these, the complete system of characteristics of the web. (Received October 28, 1933.)

267. Mr. Elihu Lazarus: *Note to Kasner's paper on the solar gravitational field.**

If we choose a metric in a six-flat $ds^2=dx^2+dy^2+dz^2+dX^2+dY^2+dZ^2$, and use the transformation equations $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$, $X=(1-2m/r)^{1/2}\sin it$, $Y=(1-2m/r)^{1/2}\cos it$, $i=(-1)^{1/2}$, $Z=\int[1-256\,m^4/(R^2+16\,m^2)^3]^{1/2}\,dR$, $R=[8m(r-2m)]^{1/2}$, we get the Schwarzschild solution with an unessential change in sign. Z is in general hyperelliptic. But if we choose $m=1/4$ it becomes elliptic: $Z=\int[1-1/(R^2+1)^3]^{1/2}dR$. Substitute $R=(x^2-1)^{1/2}$, $Z=\int[(x^4+x^2+1)^{1/2}/x^2]dx$. If we factor the expression x^4+x^2+1 and make a new substitution, the integral becomes of the form $\int\{[(1-y^2)(1-k^2y^2)]^{1/2}/y^2\}dy$, $x^2=-(b/a)y^2$. This can be integrated by parts. The final answer is $Z=i(ab')^{1/2}\cdot[-(4r^2+2r+1)^{1/2}/(i(2ar/b)^{1/2})-(1+k^2)\operatorname{sn}^{-1}i(2ar/b)^{1/2}+2(-Z(\operatorname{sn}^{-1}i(2ar/b)^{1/2})+(\operatorname{sn}^{-1}i(2ar/b)^{1/2})Z'(0)]$. (Received September 26, 1933.)

268. Dr. Abraham Sinkov: *A property of cyclic substitutions of even degree.*

A solution is given in this paper to the following problem: Is it possible to set up two permutations of the same n symbols without having at least one pair of these symbols separated by the same interval in both permutations? The answer is yes if n is odd; no if n is even. The second of these results leads to the following two theorems in abstract group theory: (1) Given any two cy-

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clic substitutions s and t of degree $2m$, and a number p, prime to $2m$, there always exists at least one x, less than $2m$, for which $s^x t^{-px}$ is of degree less than $2m$. (2) Whenever any such x is prime to $2m$, the group generated by two substitutions s and t, if non-cyclic, contains invariant subgroups generated by $2m$ conjugate substitutions of degree less than $2m$. (Received October 6, 1933.)

269. Mr. Barkley Rosser: *A mathematical logic without variables.*

By combining the results of A. Church and H. B. Curry, a system of logic is developed. This system has the following properties. (A) The postulates contain no variables (as do a number of the postulates of the *Principia Mathematica*). (B) The undefined terms are all constants and are finite in number. (C) The rules of procedure do not permit of arbitrary substitutions (as would the rule of procedure "If X is composed of undefined terms, then $X=X$"). (Received October 8, 1933.)

270. Dr. R. H. Cameron (National Research Fellow): *Algebraic functions of uniformly almost periodic functions.* Preliminary report.

Let $f_1(t), f_2(t), \cdots, f_n(t)$ be uniformly almost periodic functions of the real variable t, and let $0<h\leqq |D[f_1(t), \cdots, f_n(t)]|$ for all t, where $D(a_1, \cdots, a_n)$ is the discriminant of $z^n+a_1z^{n-1}+\cdots+a_n=0$. It is shown in this paper that under the above conditions the n continuous solutions of $[z(t)]^n+f_1(t)[z(t)]^{n-1}+\cdots+f_n(t)=0$ are all uniformly almost periodic. (Received October 5, 1933.)

271. Dr. H. W. Raudenbush: *Certain differential ideals having finite basis sets in a new sense.*

In this paper the terms differential ideal, differential field, and form are used as in the author's dissertation (Annals of Mathematics, vol. 34, pp. 509–517). Forms with coefficients in an arbitrary fixed differential field are considered. A differential ideal that contains any form that has a power in the differential ideal is called perfect. Any set of forms determines uniquely a perfect differential ideal, the intersection of all perfect differential ideals containing the forms of the set. Any perfect differential ideal is determined by a finite subset. A chain condition holds for perfect differential ideals. Any perfect differential ideal is the intersection of a unique finite set of irreducible or prime perfect differential ideals. (Received October 7, 1933.)

272. Dr. R. E. Basye: *Simply connected sets.*

A connected set M is defined to be *simply connected* if for each pair of points A and B of M, and any relatively closed subset L of M that separates A from B in M, there exists a connected subset of L which separates A from B in M. A similar definition is given for sets which are *simply connected in the weak sense.* A connected and locally arcwise connected subset M of the plane is simply connected if and only if the interior of every simple closed curve lying in M is a subset of M. Every compact plane continuum which does not separate the plane is simply connected in the weak sense. These results and others are found to have a number of applications. (Received September 26, 1933.)

273. Dr. A. B. Brown and Professor B. O. Koopman: *Structure of the Riemann multiple-space for algebroid functions.*

The authors treat the Riemann multiple-space (R. M. S.), the generalization to the case of n independent variables of the Riemann surface. The R. M. S. is proved to be a generalized manifold, and a simple geometric characterization of the latter is given. The locus of non-spherical points (points not having a neighborhood on the R. M. S. which is a $2n$-cell) is proved to be of dimension not greater than $2n-4$. That this dimension is actually attained has been shown by Osgood through an example, for the case $n=2$. The definition and some of the geometric results were announced in preliminary form in this Bulletin, vol. 33, p. 406 (abstract entitled *The Riemann multiple-space and algebroid functions*). (Received October 7, 1933.)

274. Dr. E. C. Klipple: *Two-dimensional spaces in which there exist contiguous points.*

R. L. Moore has formulated a set of axioms (A and B below, and 0, 1, and 2 of *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13) in terms of the undefined notions point, region, and "contiguous to." In the present paper, Moore's set of axioms and four additional axioms (analogues of Moore's Axioms 3, 4, and 5 of the above mentioned work) are used in proving a considerable number of fundamental theorems of plane point set theory. A considerable portion of the paper is devoted to the establishment of lemmas needed in the proof of the analogue of the plane theorem that if each of the simple closed curves J and C encloses the point P, then there exists a simple closed curve Q which is a subset of $J+C$ and whose interior contains P and is enclosed by both J and C. Axioms A and B are as follows: Axiom A. *If the point P is contiguous to the point Q, then Q is contiguous to P, and Q and P are distinct.* Axiom B. *If K is a closed point set, and H is a point set every point of which is contiguous to at least one point of K, then K contains all the limit points of H.* (Received October 6, 1933.)

275. Dr. N. E. Rutt: *Three theorems on frontiers.*

Suppose that the boundary of the connected plane domain γ is the bounded continuum Γ. The following three theorems are proved about Γ in this paper. If every proper subcontinuum of Γ is irregular, then Γ is indecomposable. If Γ is the limit sum of the collection $[D_i]$ of its subcontinua, where, for each subscript i, D_i is indecomposable and $D_i \subset D_{i+1}$, then Γ is indecomposable. Let Δ be a proper subcontinuum of Γ and $\Gamma-\Delta$ be the sum of a collection $[\Gamma_\alpha]$ of mutually exclusive sets contained in the domain δ complementary to Δ any two points of each one of which belong to a subcontinuum of it which is non-compact in δ. If the limits of the series $[\Gamma_i]$ of $[\Gamma_\alpha]$ include a point in some element of the series, then the limit sum of the series in δ is an indecomposable continuum non-compact in δ. (Received September 18, 1933.)

276. Dr. C. W. Vickery: *Spaces in which there exist uncountable convergent sequences of points.*

Spaces Λ, similar to spaces L of Fréchet except that the convergent sequences are not necessarily countable, are studied. A definition of distance applicable to such spaces is introduced, and various theorems concerning higher-type separability, compactness, and other properties are proved. Certain analogous theorems for spaces L and D of Fréchet are special cases of such theorems. It is found that all spaces which satisfy the generalized metric conditions and in which all convergent sequences of points are of type greater than ω are totally disconnected. In order to treat connected spaces a new set of axioms is introduced, the principal axiom of which is a modification of an axiom of R. L. Moore. Examples are constructed of various types of spaces considered in this treatment. (Received October 2, 1933.)

277. Professor H. T. Engstrom: *Existence theorems for relative cyclic fields.*

This paper gives a proof for the existence of an infinite number of fields which are cyclic and of given degree n with respect to an algebraic field k and in which a finite number of finite or infinite prime ideals of k prime to n have prescribed decompositions. The theorem is proved without the use of the class field theory. It is a generalization of a theorem of Hasse (Mathematische Zeitschrift, vol. 24 (1925), p. 149), who makes use of the class field theory for his construction. (Received October 7, 1933.)

278. Mr. K. S. Ghent: *A note on nilpotent algebras in four units.*

In volume 9 of the Transactions of this Society, R. B. Allen gave, without proof, a classification of all associative nilpotent algebras in $n \leqq 4$ units into non-equivalent and non-reciprocal classes of algebras. His results for $n \leqq 3$ were later verified in the master's thesis of A. A. Albert. The present author has recently re-classified nilpotent algebras in four units, and has discovered many errors in Allen's classification. He has reduced Allen's table from sixteen to nine classes, and gives a proof of the validity of the new table. (Received September 11, 1933.)

279. Professor Oystein Ore: *Contributions to the theory of higher congruences.*

The theory of higher congruences is treated from the point of view of the theory of representations, applying the results of a recent paper on p-polynomials (Transactions of this Society, July, 1933). Various new and old results are obtained by a unified method. New types of irreducible polynomials have been deduced. (Received October 5, 1933.)

280. Professor W. V. Parker: *On symmetric determinants.*

In this Bulletin, vol. 38 (1932), p. 259, the author stated and proved the following theorem: *If in a real non-vanishing symmetric determinant of order five, the elements in the principal diagonal are all zero, and the complementary minors of four of these elements are also zero, then the complementary minor of the* *remaining element is not zero.* The present paper contains a new proof of this theorem based on certain considerations from geometry. The analogous theorem for fourth-order determinants is also considered here. (Received September 20, 1933.)

281. Dr. Max Coral: *A property of self-adjoint elliptic partial differential equations.*

Let $v(x, y)$ be a non-vanishing solution in a region D of the elliptic self-adjoint equation $L(v) = av_{xx} + 2bv_{xy} + cv_{yy} + dv_x + ev_y + f = 0$ $(a_x + b_y = d,\ b_x + c_y = e,\ ac - b^2 > 0)$. If $u(x, y)$ is of class C' in D and if on every circle in D, $\int u[(av_x + bv_y)dy - (bv_x + cv_y)dx] = \int v[(au_x + bu_y)dy - (bu_x + cu_y)dx]$ then $u(x, y)$ is of class C'' in D and $L(u) = 0$. This theorem contains as special cases the generalization by Gergen and Saks of the classical Bôcher-Koebe theorem on harmonic functions. The proof is easily made by regarding $L(u) = 0$ as the Lagrange equation of a double integral variation problem and applying the Haar Fundamental Lemma and its converse in a form recently established by the writer. (Received October 6, 1933.)

282. Professor E. L. Dodd: *Means and the complete independence of certain of their properties.*

From the modern view-point, the mean $m = f(x_1, x_2, \cdots, x_n)$ of n elements, $x_1, x_2, \cdots, x_n$, has but one condition to satisfy if f is single-valued; namely: $f(x, x, \cdots, x) = x$. From this it follows that $f(m, m, \cdots, m) = f(x_1, x_2, \cdots, x_n)$. In this felicitous form, the f is determined from the *use* that is to be made of the mean. The various generalizations that a number of writers have made appear to be included in the form $\Omega[F(m)] = \Omega[F(x)]$, where m takes the place of x in $F(x)$, but not in the operator Ω to be applied to $F(x)$. This equation may be viewed as an extension of the First Theorem of the Mean in the integral calculus. Among the properties designated by internal, unique, homogeneous, translative, symmetric, increasing, monotone (increasing), associative, and continuous, the first five properties are completely independent; also the set from the second to the sixth, inclusive; also certain sets of four properties including continuity and uniqueness. (Received September 30, 1933.)

283. Professor G. C. Evans: *The Dirichlet integral and the sweeping-out process.* Preliminary report.

Let $m(e)$ be a distribution of positive mass on a bounded set, and $V(M)$ its Newtonian potential at M. Necessary and sufficient conditions are obtained for the convergence of the Dirichlet integral $D(V)$, in terms of the average approximation over spheres for $V(M)$ and in terms of $\int V(M)\, dm(e)$. The integral D is a decreasing function of the Poincaré sweeping out process, and its limiting value, by means of Kellogg's lemma, is seen to be $D(V)$, where $V(M)$ is the potential of the "limiting" distribution of mass. If F is a closed, bounded, reduced set, and $V(M)$ its conductor potential, then $(\nabla V)^2 = 0$ almost everywhere on F; there is a single distribution of positive mass on F, in total amount equal to the capacity of F, whose potential has the upper bound unity. (Received September 22, 1933.)

284. Professor J. M. Thomas: *A lower limit for the species of a Pfaffian system.*

A proof is given of the theorem that the species of a Pfaffian system is at least equal to one-half its rank. (Received October 3, 1933.)

285. Professor J. L. Walsh: *Note on the orthogonality of Tchebycheff polynomials on confocal ellipses.*

The Tchebycheff polynomials $p_n(z)$ found by orthogonalizing the set 1, z, z^2, $\cdots$ on the segment $-1 \leqq z \leqq 1$ with respect to the norm function $n(z) \equiv |1-z^2|^{-1/2}$ are also orthogonal with respect to the same norm function on every ellipse C whose foci are $+1$ and -1, in the sense $\int_C n(z) p_n(z) \overline{p_k(z)} \, |dz| = 0$, $n \neq k$. (Received September 14, 1933.)

286. Dr. Clement Winston: *On the zeros of Hermite and Laguerre polynomials.*

In this paper we consider the zeros $x_{i,n}(i=1, \cdots, n)$ of the Hermite and Laguerre polynomials defined by the relation $\int_a^\infty \phi_n(x)\phi_m(x)p(x)dx = \delta_{m,n}$, where $p(x) = e^{-x^2}$, $e^{-x}x^{\alpha-1}(\alpha>0)$, according as $a = -\infty$, 0 $(\phi_n(x_{i,n}) = 0 (i=1, \cdots, n))$. Upper and lower bounds for $x_{i,n}$ in terms of i and n, and also for the difference $x_{i+1,n} - x_{i,n}$, are found for these polynomials. (See also a paper by the author, *On the mechanical quadratures formulae involving the classical orthogonal polynomials*, soon to appear in the Annals of Mathematics.) Certain asymptotic expressions for the zeros are obtained, e.g., for the Laguerre polynomials $\lim_{n\to\infty}(x_{n,n}/(4n)) = 1 (1/2 \leqq \alpha \leqq 3/2)$. (Received October 6, 1933.)

287. Professor F. F. Decker: *The elements of the group, I, of isomorphisms of the prime power abelian group, $g = (P_1, P_2, \cdots)$, of type $(1, 1, \cdots)$, whose periods are powers of the prime.*

The author expresses the elements of the group I mentioned in the title as permutations of the elements of g. The results are obtained by the application of relations of number theory and of recurrence formulas to the exponents of the P_i's occurring in the elements of I. (Received October 9, 1933.)

288. Dr. E. J. McShane: *The analytic nature of surfaces of least area.*

Let the equations (1) $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, (u, v) on B, represent a continuous surface, the region B consisting of a Jordan curve plus its interior. If an open subset B_1 of B is such that the functions x, y, z are all constant on the boundary of B_1 but are not all constant on B_1, we say that B_1 defines an *excrescence* on the surface. If the surface (1) has a Jordan curve for boundary and has a (finite) area which is the least possible among all surfaces with that boundary, then (1) has at most a denumerable number of maximal open sets B_i defining excrescences, and these are simply connected. We remove these excrescences by giving x, y, z their constant boundary values on all of B_i. Our principal theorem is that the surface thus defined by removing excrescences is a minimal surface in the sense of differential geometry. (Received October 9, 1933.)

289. Mr. J. L. Vanderslice: *Non-holonomic geometries.*

In this paper the notion of a generalized Klein space is defined through four sets of axioms, A, B, C, D, and the general theory is developed insofar as it is independent of particular geometries. According to our definition a generalized Klein space consists of (1) an n-dimensional manifold A_n satisfying Veblen and Whitehead's axioms for differential geometry (axioms A), (2) a set of isomorphic classical Klein spaces $B_n(P)$, one associated with each point P of A_n, each B_n being characterized by the set of axioms B and the association with A_n by the set C, (3) a method (infinitesimal displacement) of establishing an isomorphic correspondence between the B_n's at different points of A_n (axioms D). A fundamental set of differential equations for the introduction of preferred coordinate systems in A_n is derived. When these equations are integrable the generalized space is locally of the same classical Klein type as the associated spaces. The general theory is then applied to the generalization of affine, projective, euclidean, and non-euclidean geometry. Among other results obtained in these special geometries there is found in all four cases a unique system of paths in A_n associated with an arbitrary point field. When the point field is that of the contact points of underlying and associated spaces, the corresponding system of paths plays the role of generalized straight lines. (Received October 9, 1933.)

290. Dr. D. C. Duncan: *The completely symmetric rational self-dual curve of order nine.*

In this paper the equation of the completely symmetric rational self-dual plane curve of order nine is derived in rectangular homogeneous coordinates, namely, $49(x^9-20x^7y^2+14x^5y^4+28x^3y^6-7xy^8)-315(x^2+y^2)^4z-45(x^7-21x^5y^2+35x^3y^4-7xy^6)z^2+2415(x^2+y^2)^3z^3-5544(x^2+y^2)^2z^5+5040(x^2+y^2)z^7-1600z^9=0$. The 7 cusps, 14 crunodes, 7 acnodes, 14 proper bitangents, 7 isolated bitangents, and 7 inflexions, all of which are real, are exhibited. The 14 collineations and 14 correlations, of which 8 are polarities, 7 by real rectangular hyperbolas, the other by an imaginary circle, under which the locus is invariant are listed. A sketch is appended depicting all the singular elements and the real polarizing conics. (Received October 9, 1933.)

291. Dr. E. J. McShane: *An existence theorem for double integral problems of the calculus of variations.*

Let K be the class of all continuous surfaces (1) $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$, (u, v) on B, having a given Jordan curve C for boundary and satisfying the following conditions: (a) for almost all constant values u_0 of u, the functions $x(u_0, v)$, etc., are absolutely continuous functions of v on the segments of the line $u=u_0$ lying in B, and analogously for almost all v_0; (b) the partial derivatives x_u, etc., are summable together with their squares over B. We denote the jacobians of (1) by X, Y, Z. It is desired to find a surface (1) which minimizes an integral (2) $F(s)=\int\int f(X, Y, Z)dudv$ in the class K. If $F(S)$ is positive quasi-regular and the boundary C is rectifiable, such a surface exists. If $f>0$ for all $(X, Y, Z)\neq(0, 0, 0)$, the assumption of rectifiability of C may be omitted, and the minimizing surface in K continues to minimize $F(S)$ in a larger class than K. (Received October 9, 1933.)

292. Dr. E. J. McShane: *Existence theorems for ordinary problems of the calculus of variations.*

Problems of the form (1) $\int F(x, y, y')dx \equiv \int F(x, y^1, \cdots, y^n, y^{1\prime}, \cdots, y^{n\prime})dx =$ minimum are studied by the device of constructing a parametric integrand $G(x, y, x', y')$ defined for $x' \geqq 0$, such that for curves $y=y(x)$ with absolutely continuous $y(x)$ the functionals $\int Fdx$ and $\int Gds$ are identical. The latter integral is investigated by methods appropriate to parametric problems, suitably modified to accord with the requirement $x' \geqq 0$. Semi-continuity of $\int Gds$ is established, and under proper hypotheses a curve $x=x(s)$, $y=y(s)$, $x' \geqq 0$, is found which minimizes $\int Gds$. Further conditions are then sought which ensure that this curve can be represented in the form $y=Y(x)$ with absolutely continuous $Y(x)$. Almost all known existence theorems for plane problems (1) are thus established, along with several new ones. For space problems the method yields decidedly stronger theorems than those in the literature. (Received October 9, 1933.)

293. Dr. E. J. McShane: *The DuBois-Reymond relation in the calculus of variations.*

Let the function $y(x)$ minimize $\int f(x, y, y')dx \equiv \int f(x, y^1, \cdots, y^n, y^{1\prime}, \cdots, y^{n\prime})dx$ in the class of all absolutely continuous functions with given end values $y(x_1)=y_1$, $y(x_2)=y_2$. It is well known that $y(x)$ satisfies the DuBois-Reymond relations (the integral form of the Euler-Lagrange equations) if the derivatives $y^{i\prime}$ are bounded. Under the hypothesis that there exist positive numbers δ, M_1, M_2 for which $|f_x(\bar{x}, \bar{y}, y')| \leqq M_1 f(x, y, y') + M_2[1+\sum(y^{i\prime})^2]^{1/2}$ whenever $|x-\bar{x}| < \delta$ and $|y^i - \bar{y}^i| < \delta$, with like inequalities for the partials $\delta f/\delta y^i$, we show that the DuBois-Reymond relations hold without restriction on y'. (Received October 9, 1933.)

294. Dr. G. B. Price: *A classification of systems of linear differential equations of the first order with constant coefficients in two variables.*

A classification of the trajectories of systems of differential equations of the type specified in the title of this paper is obtained by reducing the equations to certain normal forms by means of appropriate transformations on the dependent variables x and y. The normal forms used in the present solution are not the ones ordinarily used (for the standard treatment of this problem see Bieberbach, *Differential-gleichungen*, 3d edition, pp. 74–78). Three functions of the coefficients of the given system are found which determine completely the nature of the trajectories. The trajectories are of three types; fifteen subcases can be distinguished under type I, six under type II, and six under type III. This classification uses only the simplest analysis, and gives more complete information than the classical treatment of the problem. (Received October 9, 1933.)

295. Professor A. A. Bennett: *Seven postulates for euclidean geometry.*

In the presence of given appropriate definitions a set of seven independent postulates in terms of point, order, and ratio (of segments) is shown to suffice for euclidean geometry of one or more dimensions. The discussion is based upon the systems of Veblen (*Monographs on Topics of Modern Mathematics*) and that of H. G. Forder (*The Foundations of Euclidean Geometry*, Cambridge, 1927). (Received September 20, 1933.)

296. Professor B. H. Brown: *Equiareal maps with conic meridians and parallels.*

In this paper a complete classification is given of the area-preserving transformations in the plane which carry the rectilinear coordinate lines into a system of straight lines or conics. The classification used permits every transformation to be given as one of sixteen types, but in these types many cases special algebraically are limiting geometrically, so that the number of distinct geometric types is very much larger. When extended to the mapping of the meridians and parallels of a sphere on a plane, the six known conic projections,—the Lambert cylindrical, the Collignon, the Deetz and Adams parabolic, the Mollweide, the Lambert conical, and the Albers,—appear as very special cases. Many of the new types seem well adapted for cartographic use. (Received September 26, 1933.)

297. Professor J. M. Thomas: *An existence theorem for generalized pfaffian systems.*

Cartan's existence theorem for a pfaffian system is extended to systems whose left members are symbolic differential forms of arbitrary degrees. A necessary and sufficient condition for the existence of a non-singular integral variety on which a given set of variables are independent is also obtained in a form which is new even in the linear case. (Received October 20, 1933.)

298. Professor D. F. Gunder: *The flexure problem for rectangular beams with slits.*

Experimental and approximate theoretical studies, made at the United States Forest Products Laboratory, Madison, Wisconsin, show that the mean shearing stress in the neutral plane of a rectangular beam with symmetrical horizontal slits or checks extending along its lateral faces is less than that given by the usual engineering formulas; that is, there is a definite "two beam" action which relieves the shearing stress in the portion of the beam weakened by the slits. This paper presents an exact solution of the flexure problem for a beam of such section and confirms the main features of the distribution of shearing stress found in the earlier investigation. The solution of the flexure problem requires the solution of Laplace's equation within the plane region coincident with a cross-section of the beam, subject to certain conditions along its boundary. This solution is obtained by using the Schwartz-Christoffel transformation of a polygon onto a half-plane and solving the transformed boundary value problem. The exact solution is set up as an infinite series. To simplify the calculations an approximate solution is also obtained which is used in treating a numerical example. (Received October 16, 1933.)

299. Professor A. A. Albert: *On certain imprimitive fields of degree p^2 over P of characteristic p.*

E. Artin and O. Schreier have determined all cyclic fields of degree p, p^2 over P of characteristic p. Their latter case is but a part of the more general problem of determining all imprimitive fields $P(x)>P(u)>P$ where $P(x)$ is cyclic of degree p over $P(u)$ which is cyclic of degree p over P. This problem is solved and the results for the special case, important for a discussion of normal division algebras of degree four over P, are obtained. (Received October 11, 1933.)

300. Professor A. A. Albert: *A determination of all normal division algebras of degree 4 over F of characteristic 2.*

In this paper the author determines all normal division algebras of degree four over F of characteristic two. The results are quite different from the non-modular case in that every non-primary algebra is cyclic. It is shown that all algebras are crossed products, as in the non-modular case, and that a necessary and sufficient condition that D be cyclic is that it shall contain an inseparable quadratic sub-field. For crossed products defined by a quartic field with non-cyclic regular group it is shown that the above condition is equivalent to the property that D is non-cyclic if and only if a certain quadratic form is not a zero form. (Received October 11, 1933.)

301. Professor Arnold Emch: *Some remarkable sextic space curves.*

Among the space sextic curves of genus four are those which have the peculiarity to lie on elliptic cubic cones. A sextic of this sort has been investigated by the author some years ago (American Journal of Mathematics, vol. 45 (1923), pp. 192–207). It is the sextic intersections of a symmetric quadric and a symmetric cubic surface in S_3, invariant under the G_{24}. It is on six cubic cones whose vertices are the intersections of the sides of a quadrilateral. In this paper a sextic is considered which is invariant under a G_{120} isomorphic with the symmetric group on five variables. It lies on 10 cubic cones and is remarkable on account of the fact that its 120 tritangent-planes can be constructed, based upon the geometric properties of the group. The other possibilities considered are sextics on one, two, and three cubic cones respectively. They all have their peculiar projective properties. (Received October 20, 1933.)

302. Professor A. A. Albert: *Normal division algebras over a modular field.*

Many of the recent papers written in Germany on normal division algebras D over F have been written on the assumption that F is a *perfect* field. The only possible reason for this assumption is in case F has characteristic p, D has degree n divisible by p, so that D can possibly contain inseparable sub-fields. It is shown here that if F is a perfect modular field of characteristic p then n is not divisible by p, so that the assumption that F is perfect is entirely too strong. The author also determines all normal division algebras of degree two over F

of characteristics two, all of degree three over F of characteristic three, F not a perfect field. (Received October 11, 1933.)

303. Professor H. A. Simmons: *The first and second variations of an n-tuple integral in the case of variable limits.*

In this paper are generalized, except in one detail, all of the results which were obtained relative to a double integral in a previous article (Transactions of this Society, April, 1926). By generalizing the equations of Rodriguez (Eisenhart's *Differential Geometry*, p. 122) the author has been able to replace the curvature of a curve of the previous article by $(n-1)$ curvatures of a hypersurface, and has thus obtained a simple expansion of a functional determinant of the nth order which appears in the integrands of the fundamental integrals I', I''. In the final section of this paper are mentioned further possibilities of this method of procedure, which was originally suggested by Professor G. A. Bliss, with whom the former paper was written. (Received March 18, 1933.)

304. Mr. L. B. Robinson: *On equations in mixed differences.* Part V.

Hadamard has written that the calculus of variations is the first chapter of the functional calculus. Perhaps we can make a similar remark for equations in mixed differences. For consider the equation $u'(x)=\sum_{i=1}^{n} A_i(x)\, u(a_i x+b_i) +B(x)$. The A and B have a pole at $x=\alpha$. If certain inequalities are satisfied, the above equation has a solution which depends on one parameter. This solution divides into two parts. The first is a finite series, where each term is a repeated integral of a function uniform with the exception of certain logarithmic singular points. It is very similar to a "fonctionnelle de Gateaux." We can demonstrate the convergence of the second part with the aid of a determinant of infinite order studied by the author (this Bulletin, vol. 39, p. 356). So we establish a lien with the functional calculus. (Received October 24, 1933.)

305. Mr. Garrett Birkhoff: *The order of groups of automorphisms.*

Let G be any group, and g its order. There exists just one elementary group of order g, the order of the group of whose automorphisms may be denoted by $\phi(g)$. We prove that if ν denotes the number of distinct primes dividing g, then the number a of the distinct automorphisms of G divides $g^{\nu-1}\phi(g)$. In particular, if G is hypercentral, then a divides $\phi(g)$. (Received October 18, 1933.)

306. Professor Lennie P. Copeland: *On the theory of invariants of n-planes.*

The purpose of this paper is to construct the elements of the formal theory of invariants of factorable, quaternary quantics representing polyhedrons or n-planes. This theory is based on the extension of the theory of annihilators by means of symmetric functions. A necessary and sufficient condition is found that any function of the roots having a certain set of annihilators is a function of the determinants of the fourth order that can be formed from any four factors of the n-plane. A quaternary root-difference is defined, and the theorem proved that any homogeneous function of the root-differences of a quaternary n-plane, which is such that in all products of differences of which it consists every "root" is involved in the same number of factors, is an invariant of the form. Several special complete systems of invariants and contravariants are obtained and their geometry is studied. (Received November 1, 1933.)

307. Dr. J. L. Doob (National Research Fellow) and Professor B. O. Koopman: *The resolvent of a self-adjoint transformation.*

An integral representation is found for a function $\phi(l)$ which is analytic in the upper half of the complex l-plane, which has a not negative imaginary part there, and which satisfies the inequality $\lim\sup_{t\to\infty} |t\mathfrak{I}[\phi(it)]| < \infty$ (where $t>0$ and where $\mathfrak{I}(\xi)$ represents the imaginary part of the complex number ξ). This is applied as follows. Let T be a self-adjoint transformation defined in abstract Hilbert space $\mathfrak{H}$. Then if I represents the identical transformation, $T-lI$ has a unique inverse when $\mathfrak{I}(l)\neq 0$, which is a bounded linear transformation, defined at every element of $\mathfrak{H}$. If f, g are arbitrary elements of $\mathfrak{H}$, $(R_l f, g)$ is defined when $\mathfrak{I}(l)\neq 0$ and is analytic in the lower and upper half l-planes, satisfying the inequality $|(R_l f, g)| \leqq |f| \cdot |g|/\mathfrak{I}(l)$ there. Moreover $\mathfrak{I}(R_l f, f)$ has the same sign as $\mathfrak{I}(l)$. The result described above can then be applied to find an integral representation of $(R_l f, f)$, which is extended easily to one of $(R_l f, g)$ for arbitrary elements f, g of $\mathfrak{H}$. This is then shown to be the well known integral representation of the resolvent of a self-adjoint transformation, and the paper thus gives a simple method of obtaining this representation. (Received November 4, 1933.)

308. Dr. Ralph Hull (National Research Fellow): *A determination of all cyclotomic quintic fields.*

Let $m=q_1 \cdots q_n$, where the q's are distinct and have values chosen from all primes of the form $5h+1$ and the integer 25. Then the equations $x^2+25y^2+25z^2+125w^2=16m$, $y^2+yz-z^2=xw$ have exactly 8.4^{n-1} sets of integral solutions. Let $c_2=10m$, $c_3=5xm$, $c_4=5\{(x^2-125w^2)/4-m\}m$, $c_5=\{(x^3-625wyz)/8-xm\}m$, for an integral solution of these equations. Then every cyclotomic quintic field contains an element θ satisfying $t^5-c_2t^3-c_3t^2-c_4t-c_5=0$. For fixed m, x, y, z, and w, let $R(\theta)=R(m; x, \cdots, w)$. Then $R(m; x, \cdots, w)=R(m_1; x_1, \cdots, w_1)$ if and only if $m_1=m$ and $(x_1, \cdots, w_1)$ is one of the 8 solutions of the above equations obtained from $(x, \cdots, w)$ by changes of sign and interchanges of y and z. The solution in radicals of the quintic in t is also given. These results are obtained by means of Weber's general discussion of cyclotomic fields (*Algebra*, vol. II, §§19–28) and Hull's discussion of the number of solutions of congruences (Transactions of this Society, vol. 34 (1932), pp. 908–937). (Received November 1, 1933.)

309. Mr. P. M. Hummel: *On continued fractions of order n.*

The extension of the theory of ordinary continued fractions to a continued fraction theory of order three or higher was first considered by Jacobi. Since then, much has been written on ternary continued fractions, the most recent authors being D. N. Lehmer and P. H. Daus. The purpose of this paper is to introduce a new development of continued fractions which is equally effective when applied to continued fractions of any order. This development is effected through the use of matrices with rational integral elements of the same order as the order of the continued fraction. This is not to be confused with the work of Whittaker and Turnbull in which they associate matrices of infinite order with binary continued fractions. In this paper many of the fundamental theorems of binary continued fractions are proved with ease for continued fractions of order n. It is proved that if the partial quotients of an n-ary continued fraction are bounded, the convergents converge to the given numbers. It is further proved that every periodic n-ary continued fraction expansion converges to $n-1$ algebraic numbers belonging to the same algebraic field of degree less than or equal to n. (Received November 3, 1933.)

310. Mr. Walter Leighton and Professor H. S. Wall: *On the transformation and convergence of continued fractions.*

Let $\Delta_n = \alpha_n\delta_n - \beta_n\gamma_n$, $(n=0, 1, 2, \cdots)$, be $\neq 0$; $(\alpha_n, \beta_n; \gamma_{n-1}, \delta_{n-1}) \neq 0$, where $(p_n, q_n; r_{n-1}, s_{n-1}) = p_n(r_{n-1}x_{2n} - s_{n-1}a_{2n}) + q_n(r_{n-1}a_{2n+1} + r_{n-1}x_{2n}x_{2n+1} - s_{n-1}a_{2n}x_{2n+1})$; $\alpha_0 + \beta_0 x_1 = 1$; $\alpha_{-1} = \delta_{-1} = 1$, $\beta_{-1} = \gamma_{-1} = 0$. Then if $a_n \neq 0$, $b_{2n} = -(\alpha_n, \beta_n; \gamma_{n-1}, \delta_{n-1})/\Delta_{n-1}$, $b_{2n+1} = -\Delta_n a_{2n}a_{2n+1}/(\alpha_n, \beta_n; \gamma_{n-1}, \delta_{n-1})$, $y_{2n} = (\alpha_n, \beta_n; \alpha_{n-1}, \beta_{n-1})/\Delta_{n-1}$, $y_{2n+1} = (\gamma_n, \delta_n; \gamma_{n-1}, \delta_{n-1})/(\alpha_n, \beta_n; \gamma_{n-1}, \delta_{n-1})$, and $A_n{}^{\xi}/B_n{}^{\xi}$, $A_n{}^{\eta}/B_n{}^{\eta}$ are the nth convergents of $\xi = x_0 + [a_n/x_n]_1^{\infty}$, $\eta = y_0 + [b_n/y_n]_1^{\infty}$, respectively, the following relations hold: $S_{2n}{}^{\eta} = \alpha_n S_{2n}{}^{\xi} + \beta_n S_{2n+1}{}^{\xi}$, $S_{2n+1}{}^{\eta} = \gamma_n S_{2n}{}^{\xi} + \delta_n S_{2n+1}{}^{\xi}$, $(n=0, 1, 2, \cdots)$, where "S" stands for "A" or "B/" The authors employ this transformation in conjunction with known convergence criteria applied to η to obtain new criteria for ξ. The simple case $\alpha_0 = \delta_0 = 1$, $\beta_0 = \gamma_0 = 0$, $\alpha_n = \delta_n = 0$, $\beta_n = \gamma_n = 1$, $(n=1, 2, 3, \cdots)$, yields the following theorem: $1 + [h_n x/1]_1^{\infty}$, where $h_n \neq 0$ is a real or complex number, converges to a function with polar singularities only in every bounded region exterior to $x=0$ if $\lim_{n\to\infty} h_{2n+1} = 0$, $\lim_{n\to\infty} |h_{2n}| = \infty$. (Received October 12, 1933.)

311. Dr. S. B. Myers: *On n-dimensional differential geometry in the small and in the large.*

The problem considered here is that of obtaining relations between the local Riemannian geometry of an n-dimensional analytic Riemannian manifold and the topological properties of the manifold in the large. What topological properties of an analytic manifold can be deduced from a knowledge of the local Riemannian geometry in the neighborhood of just one of its points? Theorems giving partial answers to this question and similar questions are stated and proved here. A fundamental uniqueness theorem is that two simply connected "complete" analytic Riemannian manifolds which are continuations of the same local Riemannian element are isometric. Most of the results given here are generalizations to n dimensions of the corresponding theorems for surfaces given by H. Hopf, W. Rinow, and others. (Received November 2, 1933.)

312. Mr. J. F. Randolph: *Concerning measure and linear density properties of point sets.* Preliminary report.

In the Lebesgue theory of measure the fact that the inner measure of a point set is the upper limit of the measure of its closed components plays a central role. It has not been proved that the corresponding theorem follows from Carathéodory's five measure axioms. In the first part of the present note it is shown that a specialization of Carathéodory's fifth axiom makes the proof of this corresponding theorem possible. By this specialization none of the results derived by Carathéodory from his five axioms are lost. Furthermore, linear measure for point sets in n-dimensional space as defined by Carathéodory satisfies the modified as well as the original axioms. In the second part of the note the upper and lower Carathéodory linear density functions are shown to be Carathéodory linear measurable. From this fact some of the results of R. L. Jeffery (Transactions of this Society, vol. 35 (1933), pp. 629–647) follow directly. (Received October 10, 1933.)

313. Professor W. C. Risselman: *On approximation to the solution of a normal system of ordinary linear differential equations.*

This paper is concerned with problems of approximation on a given finite interval to the solution of the system of equations $dx_i/dt=\theta_{i1}(t)x_1+\cdots+\theta_{im}(t)x_m$, $x_i(t_0)=c_i$, $i=1,\cdots,m$. Sets of polynomials $P_{1n_1}(t),\cdots,P_{mn_m}(t)$ satisfying certain least mth power criteria are used to make the approximation. Under suitable hypotheses, questions of existence, uniqueness, and convergence are discussed. (Received November 2, 1933.)

314. Dr. W. J. Trjitzinsky: *Analytic theory of linear differential equations.*

In this paper, which will appear in the Acta Mathematica, the analytic theory of linear differential equations is completely developed for the unrestricted case of the roots of the characteristic equation and from the point of view of the asymptotic properties. (Received October 30, 1933.)

315. Dr. W. J. Trjitzinsky: *The general case of linear integro-differential equations.*

Existence of analytic solutions of the equation $L(y)=V(y)$ is established. V and L are operators of the Volterra and differential types, respectively. The treatment is for the unrestricted case of the roots of the characteristic equation, and is based on the author's theory of equations $L(y)=0$. (Received October 30, 1933.)

316. Dr. Hassler Whitney: *Functions differentiable on the boundaries of regions.*

A region R on n-space has the property A if there is a number ω with the following property. Given any two points x and x' of R whose distance apart is r, there is a curve in R joining them of length $L\leqq\omega r$. We have the following theorem: If R has the property A, $(\delta^{i_1+\cdots+i_n}/\delta x_1^{i_1}\cdots\delta x_n^{i_n})f(x_1,\cdots,x_n)=\phi_{i_1\cdots i_n}(x_1,\cdots,x_n)$ in R, and $\phi_{i_1\cdots i_n}$ is continuous in $\overline{R}=R$ plus boundary, $(i_1+\cdots+i_n=m)$, then f is "of class C^m" in $\overline{R}$ in that its definition can be extended throughout n-space so that it will have continuous partial derivatives there. (Received November 4, 1933.)

317. Dr. Hassler Whitney: *Derivatives, difference quotients, and Taylor's formula.* II.

Let $f(x_1, \cdots, x_n)$ be continuous in the region R. A necessary and sufficient condition is obtained that $(\delta^{i_1+\cdots+i_n}/\delta x^{i_1} \cdots \delta x_n{}^{i_n})f(x_1, \cdots, x_n)$ exist and equal $\phi_{i_1\cdots i_n}(x_1, \cdots, x_n)$ uniformly as $h\rightarrow 0$, $(i_1+ \cdots +i_n=m)$, where $\Delta_h{}^{i_1\cdots i_n}f(x_1, \cdots, x_n)=\sum(-1)^{i_1+\cdots+i_n-k_1-\cdots-k_n}\binom{k_1}{i_1} \cdots \binom{k_n}{i_n}\cdot f(x_1+k_1h, \cdots, x_n+k_nh)/h^{i_1+\cdots+i_n}$. The ϕ's are then continuous. This extends to n dimensions a previous result of the author. (See this Bulletin, abstract No. 39–7–219.) We may also employ Taylor's formula in finite form. (Received November 4, 1933.)

318. Dr. G. T. Whyburn: *Cyclic elements of higher orders.*

A non-degenerate maximal subset X of a compact metric space M such that every set separating X carries an essential complete r-cycle will be called an rth-order cyclic element of M and will be denoted by E_r. In this paper a study is made of the structure of such sets M in R^n with respect to the sets E_r, and results are obtained which are analogous to the known results concerning the cyclic elements of a locally connected continuum, to which the sets E_r reduce in case M is connected and locally connected and $r=0$. Concerning connectivity (Betti) numbers $p^r(X)$, it is shown that $p^r(M)=\sum p^r(E_{r-1})$ the summation being extended over all sets E_{r-1} in M. Properties of E_r-sums are developed which are analogous to those of the A-sets in locally connected continua. A number of E_r-extensible and reducible properties are proved, among them being a γ^r-local connectivity property and an hereditary γ^r-local connectivity property, paralleling similar known cyclicly extensible and reducible properties in the case mentioned above. (Received October 30, 1933.)

319. Professor R. L. Wilder: *A characterization and generalization, by internal properties alone, of those open subsets of E_3 whose boundaries are manifolds.*

An open subset D of E_n is called *i-dimensional uniformly locally connected* if for each $\epsilon>0$ there exists a $\delta>0$ such that an i-cycle of D of diameter less than δ bounds an $(i+1)$-chain of D of diameter less than ϵ. It is shown that in order that a bounded, connected, open subset D of E_3 should have a closed manifold for boundary, it is necessary and sufficient that the Betti number $p^2(D)=0$ and that D be i-dimensional uniformly locally connected for $i=0, 1$. In particular, if it be required in addition that $p^1(D)=0$, the boundary is a 2-sphere. This theorem is the 3-space analogue of the characterization of plane Jordan regions due to R. L. Moore (Proceedings of the National Academy of Sciences, vol. 4 (1918), pp. 364–370). An analogous theorem is established for the most general (bounded) open subset of E_3 that is i-dimensional uniformly locally connected $(i=0, 1)$, it being shown that the boundary in this case has, among others, the property that almost all its components are either points or 2-spheres; furthermore, there exists a duality between the 1-cycles of the interior and exterior of such a set. (Received November 2, 1933.)

320. Mr. W. C. Mitchell: *Borel-Sannia summability for double series.*

Writing $\Phi(x, y)=e^{-x-y}\sum S_{i+r-1,\ j+s-1}\ x^iy^j/i!j!$, where $i\geqq 0$, $j\geqq 0$, and $S_{m,n}=\sum u_{i,j}$, $(i, j=0, 0 \cdots m, n)$ $(=0$ for $m<0$ or $n<0)$, we say that $\sum u_{i,j}$ is summable (B, r, s), (r and s any integers), providing $\lim \Phi(x, y)=S$. A series is so summable $(r, s\geqq 0)$ when the curtailed rows, columns, and double series resulting from the removal of the terms of the sum $S_{r-1,s-1}$ are all summable by the Borel integral. The validity of certain operations with summable series follows immediately. The definition is consistent when $\sum u_{i,j}$ converges, diverges to $+\infty$, diverges to $-\infty$, provided we have respectively $|S_{m,n}|<C$, $S_{m,n}>-C$, $S_{m,n}<C$, for all m and n, where C is a positive constant. If a series is summable (B, r, s) and the first $r-1$ rows are summable (B, s), and in addition $\Phi(x, y)\rightarrow\Phi(y)$ uniformly as $x\rightarrow\infty$, and $\Phi(y)\rightarrow S$ as $y\rightarrow\infty$, then it is summable $(B, r-1, s)$. Analogous assumptions imply summability $(B, r, s-1)$. (Received November 6, 1933.)

321. Professor J. J. L. Hinrichsen: *Note on potential theory in n-space.*

In this note the elegant elementary methods developed by E. Schmidt (Bemerkung zur Potential-theorie, pp. 364–383 of Mathematische Abhandlungen H. A. Schwarz gewidmet, Berlin, 1914) and applied by him to Newtonian potentials are extended to obtain the properties near the acting masses of the integrals defining the potentials of simple surface, double surface, and volume distributions in n-space. (Received October 24, 1933.)

322. Miss Marjorie Leffler: *A lemma in potential theory.* Preliminary report.

The following lemma with the sketch of its proof has been communicated orally to the author by Professor Tibor Radó. *Lemma*: If $g(x, y)$ is a continuous real function with continuous first partials, given on the interior of the unit circle and such that at every point therein $(g_x^2+g_y^2)^{1/2}\leqq 1/\delta^{1-\lambda}$ where δ is the shortest distance from the point to the boundary, and λ is a constant, $0<\lambda<1$, then if P_1 and P_2 are any two points on the interior of the circle it follows that $|g(P_1)-g(P_2)|\leqq K\cdot\overline{P_1-P_2}^{\lambda}$ where K is a finite constant depending only on λ. The first object of this paper is to investigate the constant K. The second object is to simplify, by means of this lemma, the method of successive approximations for partial differential equations of the elliptic type as developed by Korn and others, and to apply the lemma to diverse problems in potential theory. (Received November 1, 1933.)

323. Professor J. H. Roberts: *On a problem of Knaster and Zarankiewicz.*

In 1926 Knaster and Zarankiewicz proposed the following problem (*Fundamenta Mathematicae*, vol. 8, problem 42): "Does every continuum A contain a subcontinuum B such that $A-B$ is connected?" Knaster has since shown, by an example in 3-space, that the answer is in the negative. The following question remains: "Does every plane continuum A contain a subcontinuum B such that $A - B$ is connected?" In the present paper it is shown by an example that the answer to this question also is in the negative. (Received November 3, 1933.)

324. Professor F. M. Weida: *On measures of contingency*. Preliminary report.

In his theory of contingency, Pearson (*On the correlation of characters not quantitatively measurable*, Philosophical Transactions of the Royal Society, A, vol. 195, pp 1–47) appears to use the definition of probability used in practically all treatises on the subject. This definition excludes the whole field of statistical probability. It seems fairly obvious that the development of statistical concepts is approached more naturally from a limit definition for probability than from the familiar definitions suggested by games of chance. It is the purpose of this paper to improve the treatment of Pearson's theory of contingency and make it more elegant for theoretical as well as empirical discussions. To accomplish this, use is made of the notion of characteristic function and a definition of probability that it is believed includes all forms of probability. It is believed that Pearson's conception of contingency has thus been idealized. Multiple as well as partial contingency is discussed. The author also considers briefly the case of certain dependent events and the concept of exclusiveness. (Received November 2, 1933.)

325. Dr. A. F. Moursund: *On the Nevanlinna and Bosanquet-Linfoot summation methods.*

In Part I of the paper we show that for $p=0, 1, 2, \cdots$, and (i) $\alpha=p$, $\beta>1$, (ii) $p<\alpha<p+1$, or (iii) $\alpha=p+1$, $\beta\leqq 0$, the second form of the Bosanquet-Linfoot (α, β) summation method (Journal of the London Mathematical Society, vol. 6 (1931), pp. 117–126; Quarterly Journal of Mathematics, Oxford Series, vol. 2, (1931), pp. 207–229) can be obtained from our N_{q_p} method (Annals of Mathematics (2), vol. 33 (1932), pp. 773–784; vol. 34 (1933), pp. 772–792) by specialization of the kernel $q_p(t)$; and point out that some of the theorems given by Bosanquet and Linfoot concerning summability of Fourier series and all of the theorems of Smith (Quarterly Journal of Mathematics, Oxford Series, vol. 4 (1933), pp. 93–106) follow from our corresponding N_{q_p} method theorems. In Part II we set up, by modifying the N_{q_p} method, a more general summation method (the N_{z_p} method) and obtain for the method all results concerning Fourier series given in our papers referred to above for the N_{q_p} method. (Received October 27, 1933.)

326. Professor P. H. Daus: *Ternary continued fractions in a cubic field.*

In previous papers the author has considered periodic ternary continued fraction expansions for cubic irrationalities in a ring $\mathfrak{R}(1, \theta, \theta^2)$, where θ is a root of the irreducible cubic $x^3+px^2+qx+r=0$. This paper is concerned with the analogous expansion in the cubic field $\mathfrak{K}(\theta)$, with minimal basis $(1, \omega_1, \omega_2)$. These periodic expansions lack the skew-palindromic relations previously discussed, but the expansion for $1:\omega_1:\omega_2$ is closely related to that for the direction cosines of the asymptotic line of the surface $N(x+\omega_1 y+\omega z)=m$. Omitting the non-periodic partial quotient elements p_1, q_1, p_2, and the two terminal partial quotient sets, the remaining p's and q's of one expansion are the p's and q's of the other in reverse order. In the case of a ring, previously discussed, the basis and the direction cosines lead to the same periodic elements. (Received October 30, 1933.)

327. Professor P. H. Daus: *Ternary continued fractions for cubic units.*

When periodic expansions have been found, units in the given ring or field are directly determined by the convergent sets, but the fundamental unit does not always appear. This happens when considering an equation which defines a fundamental unit ϵ, $|\epsilon|>1$. By considering the cubic equation which defines $\eta=\pm 1/\epsilon$, $(0<\eta<1)$, we obtain a periodic ternary continued fraction expansion whose convergents give all powers of ϵ in term of η. This is of use in determining all units $a+b\eta$ in the given field, which is done by finding those ϵ^k, obtained from convergent sets of a special type. In case the discriminant is positive, we obtain such an expansion corresponding to each of the three roots. (Received October 30, 1933.)

328. Professor D. N. Lehmer: *A census of squares of order* 4, *magic in the rows, columns, and diagonals.*

This is an enumeration of magic squares of order 4, magic in the rows, columns, and diagonals. Frenicle's results are checked by a method which does not depend on intricate diagrams but only on diophantine analysis. Certain automorphic transformations of these squares are used which Frenicle seems to have been unaware of. (Received October 30, 1933.)

329. Professor W. M. Whyburn: *Matrix differential systems.*

This paper is concerned with linear differential systems of the type $dY/(dx)+\sum_{i=1}^{n}A_i(x)YB_i(x)=R(x)$, where Y, A_i, B_i, and R are square matrices of m rows. Relations are established between this system and certain differential systems of the form $dU/(dx)+Q(x)U=S(x)$, where the square matrices involved are matrices of m^2 rows. The adjoint system of differential equations is set up and a number of relations between the solutions of this sytem and those of the given system (in the case $R(x)=0$, the zero matrix) are exhibited. The results of the paper are applied in a study of non-linear, Riccati type, differential systems of the form $dV/(dx)+VV=R(x)$, where V and R are square matrices. Results established by Sylvester, Wedderburn, Cayley, Hitchcock, and others for algebraic analogues of the above equation are used in the present study. (Received October 30, 1933.)

330. Professor Clifford Bell: *On the properties of a determinant function.*

The functions $H_i(t)=f_1(t)f_2'(t)f_3^{(i)}(t)$ are studied and are found to have useful applications in determining certain singularities of curves whose parametric equations, in homogeneous coordinates, are $x_1:x_2:x_3=f_1(t):f_2(t):f_3(t)$. Special attention is given to the case where $f_1(t)$, $f_2(t)$, $f_3(t)$ are rational and algebraic, for which the deficiency of the curve is zero. (Received October 31, 1933.)

331. Professor Clifford Bell: *Interpolation in the mathematics of finance.*

In an article in the American Mathematical Monthly, vol. 36 (1929), pp. 379–382, W. L. Hart has shown that the results of linear interpolation for the time from the compound interest, present value of an annuity, and amount of an annuity tables may be interpreted in such a way that the results are useful. This note is concerned with similar interpolations from the bond, the present value, the $1/a_{\overline{n}|}$, and the $1/S_{\overline{n}|}$ tables. The results are shown to have practical value as well as theoretical interest. (Received October 31, 1933.)

332. Professor A. D. Michal: *A set of postulates for "Riemannian" differential geometry in abstract vector spaces.*

This paper deals with a set of postulates for an abstract Riemannian space. The topological bearer of this space is taken to be an abstract linear vector space $S(A)$ (essentially Fréchet's *espace vectoriel D*) closed under multiplication by numbers of A, the real or complex system. On this bearer is superimposed a structure consisting of a set of associated abstract linear vector spaces and certain interspace functional transformations. In particular there is postulated a real numerically valued quadratic differential form with Fréchet differentiability properties in certain topological subspaces of $S(A)$. For an alternative set of postulates, analyticity properties are introduced in the place of mere differentiability ones. The theory of such a space is termed an abstract Riemannian differential geometry. A considerable part of certain functional differential geometries studied by the author during the last seven years can be brought under the above abstract theory in the form of special instances. (Received November 1, 1933.)

333. Professor A. D. Michal: *A theory of parallel displacement and curvature for "Riemannian" differential geometry in abstract vector spaces.*

The immediate consequences of a postulate system for an abstract Riemannian differential geometry were given by the author in another paper (see abstract 39-11-332). In the present paper the writer develops a theory of parallel displacement of an abstract vector field along a curve in the topological bearer of the geometry. The theory of two abstract objects, the correspondents of the Riemann-Christoffel curvature tensor and the Ricci tensor in classical Riemannian differential geometry, is developed systematically. Abstract Riemannian spaces of constant Riemannian curvature are then considered briefly. (Received November 2, 1933.)

334. Professor A. D. Michal: *Existence theorems for analytic solutions of ordinary, certain partial, and total differential equations in abstract vector spaces.* Preliminary report.

In this paper the author discusses the existence theorems for analytic solutions (1) of ordinary equations $dy/dt=f(y, t)$, where $f(y, t)$ is analytic on the vector space $E(A)A$ to $E(A)$; (2) of completely integrable Pfaffian equations $dy(x)=f(y, x, dx)$, where $f(y, x, dx)$ is a function on $E_2(A)(E_1(A))^2$ to $E_2(A)$ and linear in dx; (3) of certain partial differential equations. The A above denotes the real or complex number system normed with the ordinary modulus. (Received November 2, 1933.)

335. Professor A. D. Michal: *Analytical and geometrical investigations in normed linear algebras with a finite or denumerably infinite basis.* Preliminary report.

In this paper the author makes a study of normed linear abstract vector spaces with a finite or denumerably infinite basis. In particular, normed linear algebras with a finite or denumerably infinite basis are considered. Previous results of the author on abstract function theory and abstract differential geometry are here applied. Some recent function theoretic results of N. Spampinato and F. Ringleb in normed linear algebras with a finite basis are also used. (Received November 2, 1933.)

336. Professor A. D. Michal: *Abstract dynamical systems and contact transformations.* Preliminary report.

A set of postulates is given for Lagrangean and Hamiltonian systems in abstract vector spaces. Various analytic and geometric results obtained by the author in previous papers are here used. Kerner-Graves integration theory in vector spaces plays a role in the abstract variational principles. (Received November 2, 1933.)

337. Professor A. D. Michael: *Linear connections and non-Riemannian differential geometry in abstract vector spaces.*

The methods developed by the author for abstract Riemannian geometries in abstract vector spaces are here extended to a theory of non-Riemannian spaces in abstract spaces. The author discusses the theory of parallel displacement and curvature arising from the consideration of one or several abstract linear connections. (Received November 2, 1933.)

338. Professor A. D. Michal: *An definitions of abstract polynomials and analytic functions in abstract vector spaces.*

In 1929, M. Fréchet initiated the study of abstract polynomials in certain general abstract vector spaces. A more satisfactory theory for triangularly normed abstract vector spaces was more recently developed by members, notably by R. S. Martin, of the author's mathematical seminars at the California Institute. This led to a theory of abstract analytic functions. The present paper gives new definitions of abstract polynomials and analytic functions and brings the subject closer to the classical theory. (Received November 2, 1933.)

339. Professor A. D. Michal: *A critique of the postulate systems for abstract vector spaces and abstract Hilbert space.*

This note gives a careful ennumeration of the postulate systems for normed linear abstract vector spaces, and in particular for abstract Hilbert spaces (real or complex). The postulates for more general abstract spaces are then discussed. In some of the generalizations the number system is taken to be a normed topological field. (Received November 2, 1933.)

340. Professor Harry Bateman: *A partial differential equation connected with the functions of the parabolic cylinder.*

A harmonic partial differential equation with particular solutions of type $H_m(x)H_{n-m}(y)$ when n is a positive integer possesses also a solution of type $H_n(x \cos \alpha + y \sin \alpha)$ where α is a constant and $H_n(z)$ is the Hermite polynomial of order n; it can thus be used to obtained an expansion of this function. The corresponding case in which n is not an integer leads to some properties of the functions of the parabolic cylinder. These and some further properties of the functions in question are obtained partly with the aid of definite integrals. (Received November 2, 1933.)

341. Professor E. T. Bell: *Exponential polynomials.*

Three types of polynomials are generated from exponential functions as follows: $D_t^n \exp (xt^r) \equiv \zeta_n(x, t; r) \exp (xt^r)$, ($r$ integer>0); $\exp (f(t)) \equiv \exp (\alpha_1 t + \alpha_2 t^2/2 + \cdots) \equiv \phi_0 + \phi_1 t + \phi_2 t^2/2! + \cdots$; $\exp (xt^r) \exp (f(t)) \equiv \Psi_0 + \Psi_1 t + \cdots$; where n is a non-negative integer. If $r > 2$, the ξ satisfy no linear differential equation with polynomial coefficients in x and t which is of order independent of n. For $r=2$, the polynomials are Hermite polynomials. From the ξ is generated a set of functions orthogonal in the interval $-\infty$ to $+\infty$. The polynomials ϕ generalize the Appell polynomials; they include the ξ as a special case. These polynomials have numerous interesting arithmetical properties. The Ψ generalize Appell polynomials in another direction. When $r > 1$ the procedure by which Appell polynomials are linked with linear differential equations is not applicable, so that here new methods are devised. The existence of solutions of the equations follows from arithmetical considerations, and the form of the equations is determined by the integer r. When $r=1$ (Appell's case) all the special features disappear or become trivial. (Received November 2, 1933.)

342. Professor Glenn James: *On the second case of Fermat's last theorem.*

In a previous paper, to appear in the American Mathematical Monthly, the writer has determined for the "First Case" certain lower limits for the parameters entering in Fermat's equation, $x^n + y^n = z^n$. The present paper extends this work to the "Second Case." Among the results obtained is the fact that the least of x, y, z exceeds n^{n-2}, and that the difference of the two largest exceeds 2^n. The latter result removes the difficulties that have heretofore been met in proving that x, y, z are composite. (Received November 2, 1933.)

343. Professor W. F. Osgood. *On certain methods in dynamics.*

The integral $\int(\sum_r p_r \delta q_r - H\delta t)$, taken over a closed path, is known to be a relation integral invariant of a Hamiltonian system. The paper treats the inverse problem:—Let the above integral be a relative integral invariant; to determine when the system will be Hamiltonian. (Received November 4, 1933.)