ON THE SUMMABILITY AND GENERALIZED SUM OF A SERIES OF LEGENDRE POLYNOMIALS*

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1. *Introduction.* The results obtained in this paper are as follows.

(A) *The series of Legendre polynomials* $\sum n^p X_n(x)$, *where* p *is a positive integer, is summable* (H, p) *for* $-1<x<1$, *and summable* $(H, p+1)$ *for* $-1\leqq x<1$.

(B) *The generalized sum over the range* $-1<x<1$ *is*

$$\sum_1^\infty n^p X_n(x) = -\frac{1}{2(2y)^{p-1/2}}\begin{vmatrix} 2 & 0 & 0 & 0 & \cdots & 0 & 1 \\ y & 2y & 0 & 0 & \cdots & 0 & 1 \\ A_2^3 & y & 2y & 0 & \cdots & 0 & 1 \\ A_3^4 & A_2^4 & y & 2y & \cdots & 0 & 1 \\ \cdot & \cdot & & & & \cdot & \cdot \\ \cdot & \cdot & & & & \cdot & \cdot \\ \cdot & \cdot & & & & \cdot & \cdot \\ A_{p-2}^{p-1} & A_{p-3}^{p-1} & \cdots & y & & 2y & 1 \\ A_{p-1}^{p} & A_{p-2}^{p} & \cdots & A_2^p & & y & 1 \end{vmatrix}$$

where

$$y = 1 - x; \qquad A_t^p = {}_pC_t + (-1)^t{}_{p-1}C_t; \qquad (p > 2).$$

2. *The Cases* $p=0, 1, 2$. We first obtain these results for $p=0$, 1, 2. Let p be a positive integer, $S_{n,p}$ the sum of the first n terms of the series $\sum n^p X_n(x)$, $S_{n,p}^{(p)}$ the pth Hölder mean of $S_{n,p}$, and $S^{(p)}$ the limit of this mean for $n\to\infty$.

The generating function of the Legendre polynomials gives us at once the sum of the convergent series

$$(1) \qquad \sum_1^\infty X_n(x) = S^{(0)} = [2(1-x)]^{-1/2} - 1, \qquad (-1 < x < 1).$$

We can readily find $S^{(1)}$ by use of the recursion formula

$$(2) \qquad (2m+1)xX_m = (m+1)X_{m+1} + mX_{m-1},$$

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which we write in the form

$$2mxX_m = (m+1)X_{m+1} + (m-1)X_{m-1} + X_{m-1} - xX_m.$$

Summing from $m=1$ to $m=n$ we obtain, after some reductions,

$$2(x-1)S_{n,1} = (1-x)S_{n,0} + X_0 - X_1 + (n+1)X_{n+1} - nX_n - X_n.$$

The first mean now gives

$$2(x-1)S_{n,1}^{(1)} = (1-x)S_{n,0}^{(1)} + (1-x) + [(n+1)X_{n+1} - X_1 - S_{n,0}]/n.$$

For $n\to\infty$ we then have

$$(1') \qquad 2(x-1)S^{(1)} = (1-x)S^{(0)} + (1-x).$$

To obtain $S^{(2)}$ we multiply (2) by m and arrange in the form

$$2m^2xX_m = u(m+1)X_{m+1} + v(m-1)X_{m-1} - mxX_m,$$

where u and v are quadratic functions of the indicated arguments. Summing and taking two successive mean values we obtain after some reductions

$$2(x-1)S_{n,2}^{(2)} = (1-x)S_{n,1}^{(2)} + S_{n,0}^{(2)} + 1 - (1/n)\sum_{r=1}^{n}[3S_{r,1} - r(r+1)X_{r+1} + S_{r,0}]/r.$$

The last term vanishes for $n\to\infty$. We note also that the existence of the limit of any mean ensures the existence of the same limit for the higher means. Hence for $n\to\infty$ the preceding equation becomes

$$(1'') \qquad 2(x-1)S^{(2)} = (1-x)S^{(1)} + S^{(0)} + 1.$$

3. *Proof of* (A) *and* (B). We shall now show that

$$(3)\ 2(x-1)S^{(p)} = (1-x)S^{(p-1)} + \sum_{t=2}^{p-1} A_t^p S^{(p-t)} + S^{(0)} + 1, \ (p>2);$$

provided that $S^{(r)}$ exists for $r=0, 1, 2, \cdots, p-1$, and A_t^p, expressed in terms of binomial coefficients, is

$$A_t^p = {}_pC_t + (-1)^t{}_{p-1}C_t.$$

Assuming the existence of the means of order less than p we first prove the following lemma.

LEMMA. *The means of order p of n^pX_n and of $(n+1)^pX_{n+1}$ have the limit zero for $n\to\infty$, when $-1<x<1$.*

To prove this we multiply (2) by m^{p-1}, express the coefficients of X_{m+1} and X_{m-1} on the right in powers of $m+1$ and $m-1$, respectively, sum from $m=1$ to $m=n$, and write the result in the form

$$(4)\qquad 2(x-1)S_{n,p} = (1-x)S_{n,p-1} + \sum_{t=2}^{p-1} A_t^p S_{n,p-t} + S_{n,0} + X_0 + n^{p-1}(n+1)X_{n+1} - (n+1)^pX_n.$$

Successive application of this formula enables us to write

$$2(x-1)S_{n,p} = w_n + u_p(n+1)X_{n+1} + v_p(n)X_n,$$

where $w_n = A_pS_{n,0}+B_pX_0$, A_p and B_p being independent of n, and $u_p(n+1)$, $v_p(n)$ are polynomials of degree p in the indicated arguments. Denoting the mean of order k of n^pX_n by $M_{n,p}^{(k)}$, we have

$$M_{n,p}^{(1)} = S_{n,p}/n,$$

which, by use of the preceding equation, we may write in the form

$$2(x-1)M_{n,p}^{(1)} = w_n^{(1)} + u_{p-1}(n+1)X_{n+1} + v_{p-1}(n)X_n,$$

where $w_n^{(1)} = (w_n + C_pX_{n+1}+D_pX_n)/n$, C_p and D_p being independent of n, and where the coefficients of the last two terms are polynomials of the indicated degree and arguments. We note that $w_n^{(1)}$ and all of its means$\to 0$ when $n\to\infty$. Proceeding in this way we have

$$2(x-1)M_{n,p}^{(2)} = w_n^{(2)} + u_{p-2}(n+1)X_{n+1} + v_{p-2}(n)\, X_n,$$

and finally

$$2(x-1)M_{n,p}^{(p)} = w_n^{(p)} + u_0(n+1)X_{n+1} + v_0(n)X_n,$$

where each $w_n^{(r)}$ and its means$\to 0$ when $n\to\infty$. The means of $(n+1)^pX_{n+1}$ may be treated similarly.

COROLLARY 1. *The mean of order p of $u_p(n+1)X_{n+1}$ and of $v_p(n)X_n$ approach zero when $n\to\infty$, for $-1<x<1$.*

COROLLARY 2. *For $x=-1$ the mean of order $p+1$ of each of the preceding expressions vanishes when $n\to\infty$. This follows from $X_n(-1)=(-1)^n$.*

Now (3) is obtained at once by taking the limit of the mean of order p of (4). But the values of $S^{(0)}$, $S^{(1)}$, $S^{(2)}$ already found show that (3) holds for $p=3$. Hence it holds for positive integral values of $p>2$.

The result under (B) is obtained by expressing each $S^{(r)}$, $(r=1, 2, \cdots, p)$, in terms of the sums of lower order by use of (1′), (1″), (3) and solving this system of equations for $S^{(p)}$.

When $x=-1$, Corollary 2 shows that the series $\sum(-1)^n n^p$ is summable $(H, p+1)$ and a new form is obtained for its sum by putting $y=2$ in the formula under (B).

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ERRATA

This Bulletin, volume 37 (1931), pages 759–765:

On page 759, line 12, *for* $P_1\equiv P, P_2, \cdots$ *read* $P\equiv P_1, P_2, \cdots$.

On page 764, line 9, second parenthesis, *for* (x_1x_2, x_2, x_1x_4) *read* (x_1x_2, x_2^2, x_1x_4).

On page 764, line 8 from the bottom, second parenthesis, *for* $(z_2, z_3, \epsilon_3 z_4)$ *read* $(\epsilon z_2, z_3, \epsilon^3 z_4)$.

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This Bulletin, volume 39 (1933), p. 589:

Lines 13–14, omit the words: *one point of inflexion*; and add the sentence: *A point of inflexion lies at infinity on each bisector of the angles formed by adjacent cuspidal tangents.*

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