ON NATURAL FAMILIES OF CURVES

BY C. H. ROWE

1. *Introduction.* A *natural family* of curves in a Riemannian space V_N of N dimensions is a family which consists of the ∞^{2N-2} extremals of an integral $\int \mu ds$, where μ is a function of position and ds is the element of length. If a system of ∞^M curves is given in V_N, where $M \leqq 2N-2$, there does not in general exist a natural family to which all these curves belong; and the present paper is concerned with the problem of finding conditions for the existence of such a family. This problem is equivalent, as will be seen, to that of finding conditions for the possibility of representing V_N conformally on a second Riemannian space so that the curves of the second space that correspond to the curves of the given system are geodesics.

In the case where $M=2N-2$ and $N>2$, a condition is given by the extensions to Riemannian space of the so-called* theorem of Thompson and Tait and the converse theorem of Kasner. According to these theorems, a system of ∞^{2N-2} curves in V_N ($N>2$) is a natural family if and only if the ∞^{N-1} curves of the system that cut an arbitrary V_{N-1} normally are the orthogonal trajectories of a family of a single infinity of V_{N-1}'s, or in other words, form a normal congruence.† This result is applicable only when the given system contains ∞^{2N-2} curves; and it is not valid for $N=2$, since the condition is satisfied by an arbitrary system of ∞^2 curves on a surface. The result that we shall obtain in what follows is not subject to these limitations, and it yields immediately the theorem of Thompson and Tait in the cases where this is applicable.

Our argument will be mainly synthetic in form; and we shall make no attempt at a rigorous discussion of the minimum assumptions under which our results hold, contenting ourselves with supposing that the functions that we introduce, explicitly

* J. A. Schouten (Nieuw Archief, (2), vol. 15 (1928), p. 97) points out that this theorem was first given by Lipschitz.

† See E. Kasner, Transactions of this Society, vol. 11 (1910), p. 121, J. A. Schouten, loc. cit., and W. Blaschke, Nieuw Archief, (2), vol. 15 (1928), p. 202. The methods of the present paper are similar to those used by Blaschke.

or implicitly, have whatever degree of regularity is required in order to justify our reasoning, and restricting ourselves whenever it is necessary to a sufficiently limited region of V_N.

2. *Conditions for Belonging to a Natural Family.* When we consider a system (S) of ∞^M curves, we shall suppose that at least one curve of (S) passes through each point of V_N, thus excluding the cases in which all the curves of (S) lie in a sub-space; and we shall suppose that at most one curve of (S) passes through a given point in a given direction. We thus have

$$N - 1 \leqq M \leqq 2N - 2.$$

We shall further assume (when $M > N-1$) that, if two curves of (S) passing through a point are given, it is possible for a variable curve of (S) which passes through this point to move continuously from coincidence with one of these curves into coincidence with the other. When $M = N-1$, the system of curves is called a congruence; and we shall then assume that a unique curve of (S) passes through each point. When $M = 2N-2$, we shall suppose that a unique curve of (S) passes through each point in each direction.

Suppose for the present that $N > 2$. By a *surface of the system* (S) we shall mean a surface (or V_2) that can be generated by a variable curve of (S) which moves with one degree of freedom; and we shall call the ∞^1 positions of the variable curve the *generators* of the surface.* If the initial and final positions of the generating curve coincide, we shall call the surface a *tube*. We shall have to consider the curves on a surface of (S) that cut the generators orthogonally, and we shall call these the *orthogonal curves* of the surface.

When a surface of (S) is a tube, it may happen that the orthogonal curve that starts from a point P returns to P after encircling the tube, and is thus a closed curve; but in general this will not happen. Instead, if an orthogonal curve is prolonged indefinitely in either sense, it will encircle the tube continually, never intersecting itself, in a manner not unlike that in which a circular helix encircles the circular cylinder on which it lies.

* If a surface can be generated in more than one way by curves of (S), it will always be clear that a particular mode of generation is being considered.

The theorem that we wish to prove may now be stated as follows.

In order that there should exist a natural family in V_N to which the curves of a given system belong, it is necessary and sufficient that, whenever one orthogonal curve on a tube of the system is closed, all the remaining orthogonal curves on that tube should also be closed.

This theorem holds without essential change when $N=2$, although it has been expressed in language appropriate to three or more dimensions. In the case where $M=N=2$, the theorem may be stated, perhaps more suitably, by saying that a system of ∞^2 curves on a surface is a natural family if, and only if, the ∞^1 curves of the system that cut an arbitrary closed curve normally are the orthogonal trajectories of a family of ∞^1 closed curves.

The case where $N=2$, $M=1$ may be conveniently taken with the case where, for $N>2$, the system is a normal congruence. In the former case we may regard the condition of our theorem as being fulfilled. In the latter case it is always fulfilled, since every tube of a normal congruence has closed orthogonal curves. In each of these cases, as we shall see, it is possible to find a natural family containing the given system, but, in contrast to the remaining cases, this family is not unique.

That our theorem yields the theorem of Thompson and Tait follows from the fact that all the tubes belonging to a congruence have closed orthogonal curves when the congruence is normal, and in no other cases.* Suppose that a system (S) of ∞^{2N-2} curves satisfies our condition, and consider the congruence formed by the curves of (S) that cut a given V_{N-1} normally. On any tube of this congruence, one, and therefore every, orthogonal curve is closed; and consequently the congruence is normal. Conversely, supposing that the system (S) satisfies the con-

* Consider a congruence (C), and suppose that every orthogonal curve on every tube of (C) is closed. On a fixed curve C_0 of (C) take a fixed point P_0, and let C be a variable curve of (C). On any surface of (C) of which C_0 and C are generators draw the orthogonal curve that passes through P_0, and let it meet C in P. The point P does not depend on our choice of this surface, for the tube formed by two such surfaces has closed orthogonal curves. As C varies in (C), P describes a V_{N-1} which clearly cuts all the curves of (C) normally. If we vary P_0 along C_0, we get a family of a single infinity of V_{N-1}'s which cut the curves of (C) normally.

dition of Thompson and Tait, consider a tube of (S) on which one orthogonal curve is closed. If we draw a V_{N-1} through this curve cutting the generators of the tube normally, the tube belongs to the congruence formed by the curves of (S) that cut this V_{N-1} normally; and since this congruence is normal, all the orthogonal curves of the tube are closed.

3. *Proof of Necessity*. The necessity of the condition of our theorem is easily proved. Let the metric of V_N be defined in any system of coordinates x^i by the formula

$$ds^2 = g_{ij}dx^i dx^j,$$

and consider the natural family (S) formed by the extremals of $\int \mu ds$, where μ is a function of the variables x^i, which we shall suppose not to vanish or become infinite in the region under consideration. Consider a second Riemannian space V_N', in which the element of length ds' is given by

$$ds'^2 = \mu^2 g_{ij}dx^i dx^j = \mu^2 ds^2.$$

The correspondence between these two spaces in which corresponding points have the same coordinates is conformal. It is therefore sufficient to prove that the system (S') in V_N' that corresponds to (S) satisfies the condition of our theorem. Now the curves of (S'), being extremals of $\int ds'$, are geodesics of V_N'; and the presence of one closed orthogonal curve on a tube generated by geodesics implies that all the orthogonal curves of this tube are closed. This is an immediate consequence of the theorem of Gauss that two orthogonal trajectories of a system of ∞^1 geodesics on a surface intercept equal arcs on all these geodesics.

4. *A Lemma*. Before considering the sufficiency of our condition, we must establish a preliminary result. We shall show that, if (S) is a system of curves which satisfies our condition, and if P_0 and P are two given points, it is possible to find a surface of (S) which has an orthogonal curve passing through P_0 and P, except in the case where (S) is a normal congruence and in the case where $N=2$, $M=1$.

Since this result is immediate when $N=2$, $M=2$, we shall suppose that $N>2$. Consider first the case where (S) is a congruence which is not normal. Let S_0 and S be the curves of (S) that pass through P_0 and P respectively, and let Σ be any surface of (S) of which these curves are generators. Let the orthogonal curve on Σ that passes through P_0 meet S in a point Q, which we may suppose not to coincide with P. We can find a tube of (S) which contains the curve S as a generator, and which has no closed orthogonal curves; for if not, the orthogonal curves of every tube of which S is a generator would be closed, and (S) would therefore be a normal congruence. Let T be such a tube, and let the orthogonal curve P_0Q that we are considering on Σ be prolonged beyond the point Q by allowing it to encircle the tube T a certain number of times, until it cuts S in a point Q' which lies on the side of P remote from Q. This is clearly possible, if we allow the curve to encircle T a sufficient number of times in the right sense; for if not, the curve would cut S in an unending sequence of points lying between P and Q. These points would have a limiting point, and, for reasons of continuity, the orthogonal curve of T through this limiting point would be closed. Supposing that P lies between Q and Q', we shall allow the tube T to vary continuously, always passing through S, and to contract until it reduces to the curve S. While this happens, the point Q' moves continuously along S, and ultimately coincides with Q. At some stage in this process Q' must pass through P; and at this stage we have a tube which has an orthogonal curve joining Q to P. The surface made up of the portion of Σ between the curves S_0 and S and of the tube T has thus an orthogonal curve P_0QP which passes through P_0 and P, as is required.

Suppose now that (S) contains ∞^M curves, where $M > N-1$. It will easily be seen that we can form with curves of (S) a congruence (C) which is not normal, and which, like (S), satisfies the condition of our theorem. We can then find a surface belonging to (C), and therefore to (S), which has an orthogonal curve joining the two given points.

If, however, the system (S) is a normal congruence, or if $N=2$, $M=1$, it will not be possible to find a surface of (S) satisfying our requirement, unless one of the V_{N-1}'s that cut the curves of (S) normally passes through the two given points.

5. *Proof of Sufficiency.* We shall suppose that on any tube of a system (S) either all or none of the orthogonal curves are closed, and we shall show that we can define a function μ of position throughout V_N such that the curves of (S) belong to the

natural family formed by the extremals of $\int \mu ds$. We shall state the proof for $N>2$, without troubling to indicate the merely verbal changes that would be desirable if N were equal to 2.

We shall suppose firstly that (S) is not a normal congruence or a system of ∞^1 curves on a surface. Take a fixed point P_0, and choose an arbitrary positive value $\mu(P_0)$ for the function μ at P_0. Let P be any point, and find a surface Σ of (S) which has an orthogonal curve joining P_0 to P. Consider on Σ a second orthogonal curve, which we shall allow to tend to coincide with the first. Let the lengths of the arcs that these curves intercept on the generators of Σ that pass through P_0 and P be δs_0 and δs. We may assume that, as the second orthogonal curve tends to the first, the ratio $\delta s_0/\delta s$ tends to a limit; and we shall show that this limit depends only on the point P (the point P_0 being fixed), and not on our choice of the surface Σ. We must thus show that this limit is unaltered if we replace Σ by another surface Σ' which also has an orthogonal curve joining P_0 to P. If (S) is a congruence, the surfaces Σ and Σ', having in common the curves of the congruence that pass through P_0 and P, form a tube; and since the orthogonal curves of this tube are closed, the truth of our assertion is clear. In the general case $(M>N-1)$ we shall construct a tube of which Σ and Σ' form part. We connect the generators of Σ and Σ' that pass through P_0 by a surface of (S) whose generators all pass through P_0; and, similarly, we connect the generators of Σ and Σ' that pass through P by a surface of (S) whose generators all pass through P. These two new surfaces, together with Σ and Σ', form a tube of (S) on which one, and therefore every, orthogonal curve is closed. Let the second orthogonal curve that we have taken on Σ be continued around this tube until it closes. We thus get a second orthogonal curve on Σ'. If $\delta s_0'$ and $\delta s'$ are the lengths of the arcs that the two orthogonal curves on Σ' intercept on the generators of Σ' through P_0 and P, we have to show that

$$\lim \frac{\delta s_0'}{\delta s'} = \lim \frac{\delta s_0}{\delta s} \cdot$$

We notice that, if a surface is generated by curves passing through a fixed point, the arcs intercepted on two of these curves between the fixed point and an orthogonal trajectory of the curves have a ratio which tends to unity as the lengths of the arcs tend to zero. We thus have the equations

$$\lim \frac{\delta s_0'}{\delta s_0} = 1, \quad \lim \frac{\delta s'}{\delta s} = 1,$$

from which the required conclusion follows at once.

Since $\lim \delta s_0/\delta s$ depends only on the point P, we may define the value $\mu(P)$ of the function μ at any point P by the equation

$$\mu(P) = \mu(P_0) \lim \frac{\delta s_0}{\delta s},$$

which may also be written in the equivalent form

$$\mu(P)ds = \mu(P_0)ds_0.$$

It is clear from this that, if we take any surface of (S), and consider the arcs intercepted on two of its generators by two of its orthogonal curves, the values of $\int \mu ds$ arising from these two arcs are equal. We shall suppose that the function μ does not vanish or become infinite in the region that we are considering, and we shall introduce, as we did in §3, a second Riemannian space V_N' in conformal correspondence with V_N, the element of length ds' in V_N' being given by $ds' = \mu ds$. The system (S') in V_N' that corresponds to (S) has the property that the arcs intercepted on the generators of a surface of (S') by two of its orthogonal curves are all equal. In virtue of the converse of the theorem of Gauss that we quoted in §3, every curve of (S') is therefore a geodesic of any surface of (S') of which it is a generator. This implies that the first curvature-vector of a curve of (S') either vanishes or is normal to every surface of (S') of which the curve is a generator. The latter alternative is clearly impossible, and therefore every curve of (S') is a geodesic of V_N'. It follows that the curves of (S) belong to the natural family formed by the extremals of $\int \mu ds$.

Since we have defined the function μ uniquely except for a constant factor, our procedure leads to a unique natural family. It will be seen that this is in fact the only natural family to which all the curves of (S) belong.

It remains to consider the case where (S) is a normal congruence, and the case where $N=2$, $M=1$. In these cases the condition of our theorem is always verified, as we have already remarked, and we shall show that we can find a natural family which contains the given curves; but this family is no longer unique. If we try to use the same method as before, we find that we can define the value of μ only at points of the normal V_{N-1} that passes through P_0. Instead, therefore, we take an arbitrary curve which meets each normal V_{N-1} once, and we define the function μ arbitrarily along this curve. We then use our former method to define the value of μ at any point P in terms of the assigned value of μ at the point where the normal V_{N-1} that passes through P meets this curve. Reasoning as before, we can then prove that the curves of (S) are extremals of $\int \mu ds$. We can thus find a natural family depending on an arbitrary function of one variable which contains the curves of the given system.*

We may remark, in conclusion, that a modification of the methods that we have used will allow us to consider a more general problem. If we are given a system (S) of curves in V_N and a transversality relation, we may ask whether there is a first order problem of the calculus of variations such that the family of its ∞^{2N-2} extremals contains the curves of (S), and such that its transversality relation coincides with the given one. If we replace the relation of orthogonality by the given transversality relation, we are led to consider on any surface of (S) the curves that are cut transversally by the generators, and we may call these the transversal curves of the surface. It is then possible to generalize the theorem of Kneser on transversals and its converse† in the same way as our previous result gener-

* It is perhaps worth noticing that, in the cases at present under consideration, a natural family containing the given system (S) is uniquely determined if we are given one curve Γ which belongs to the natural family, but not to (S), and which cuts each normal V_{N-1} once, but not normally. We recall the fact that the first curvature-vector of an extremal of $\int \mu ds$ is the component normal to the curve of the gradient of log μ. This gradient is determined at each point of Γ, because we know its components normal to each of two distinct directions, namely, that of Γ and that of the curve of (S) through the point. The value of μ is therefore determined along Γ except for a constant factor, and consequently the natural family is determined uniquely. It will be seen that, when $N>2$, the curve Γ may not be given arbitrarily.

† See J. Douglas, Transactions of this Society, vol. 29 (1927), p. 401, and W. Blaschke, loc. cit.

alizes the theorem of Thompson and Tait. We can prove, in fact, that a condition for an affirmative answer to our question is that, on any tube of (S), either all or none of the transversal curves should be closed.

TRINITY COLLEGE,
DUBLIN, IRELAND

ON THE CONDITION THAT TWO ZEHFUSS MATRICES BE EQUAL

BY D. E. RUTHERFORD

1. *Introduction.* In a recent paper* Williamson has considered matrices whose sth compounds are equal. The present paper considers the somewhat analogous problem of finding the conditions that two Zehfuss matrices be equal.

Suppose that R is a matrix of n_1 rows and m_1 columns whose ijth element is r_{ij}, and that P is another matrix of n_2 rows and m_2 columns. Now, if the matrix Q of n_1n_2 rows and m_1m_2 columns can be partitioned into submatrices each of n_2 rows and m_2 columns such that the ijth submatrix is $r_{ij}P$, then Q is a *Zehfuss matrix*† or the *direct product matrix*‡ of R and P. We shall write

$$Q = R\langle P\rangle = \langle P\rangle R.$$

In general, however, $R\langle P\rangle \neq \langle P\rangle R$.

It is the purpose of this paper to find out under what conditions the matrix equation

$$A\langle B\rangle = C\langle D\rangle$$

is true. That is, we shall find the most general form of the matrices A, B, C, D when the above equation holds.

2. *The Simplest Case.* We shall begin by considering the simplest case, where A, B, C, D are row vectors, where A and D are of order m_1, where B and C are of order m_2, and where

$$(m_1, m_2) = 1;$$

that is to say, m_1 and m_2 are prime to one another. Suppose that

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* J. Williamson, this Bulletin, vol. 39 (1933), p. 109.

† G. Zehfuss, Zeitschrift für Mathematik und Physik, vol. 3 (1858), p. 298.

‡ L. E. Dickson, *Algebras and Their Arithmetics*, p. 119.