SIMPLIFICATION OF THE SET OF FOUR POSTULATES FOR BOOLEAN ALGEBRAS IN TERMS OF REJECTION*

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1. *Introduction.* Some time ago, I presented† a set of four postulates for Boolean algebras expressed in terms of Sheffer's "stroke" operation, or the operation of "rejection." This set, which is a reduction of Sheffer's set of five postulates for Boolean algebras,‡ uses the stroke as the only primitive idea, besides that of *class*, and retains the characteristic of Sheffer's set of defining in terms of its primitives *all* the special Boolean elements, *zero*, the *whole*, and the *negative* of an element. It is fitting that this economical set of postulates should be as simple as possible. But this is not the case. In my effort to attain economy in the number of postulates, I paid too little attention to the matter of simplicity in statement of the postulates, with the result that one of the postulates, Postulate P_4, is unnecessarily complex. It is my object now to offer a simplification of Postulate P_4.

The simplification, it will be found, will retain all the advantages possessed by the older set. It consists merely in replacing P_4 by a proposition in which the negative elements are considerably fewer, and are more symmetrically distributed, than in P_4.

In order to prove the sufficiency of the new postulates for Boolean algebras, it will of course suffice to show that my former set can be derived from them. This derivation I obtain. However, I also derive from the new postulates Sheffer's set and the well known Whitehead-Huntington set.§ Since the sufficiency of my former set is proved by showing that it yields Sheffer's set, and since the sufficiency of Sheffer's set is proved by showing that this set yields the Whitehead-Huntington set, I thought that it might be of interest to derive the latter two sets directly from my new set. These derivations, at the same time, will exhibit the workability of the new set.

* Presented to the Society, March 18, 1933.

† Transactions of this Society, vol. 17 (1916), pp. 50–52.

‡ See the Transactions of this Society, vol. 14 (1913), pp. 481–488.

§ See the Transactions of this Society, vol. 5 (1904), pp. 288–309.

I furnish for the new postulates their *complete existential theory* (in the sense of E. H. Moore), which theory will carry with it the proofs of the *consistency* and of the *independence* of the postulates. The proof-systems for this theory will be found to be of a simple arithmetic type.

2. *The New Postulates.* The new postulates have as undefined ideas the undefined ideas of the old set, namely, a *class* K and a *binary operation* $|$. The postulates are the propositions Q_1–Q_4 below. In Postulates Q_3 and Q_4, there is to be understood the condition *if the elements involved and their indicated combinations belong to* K. The supplying of this condition is essential in the consideration of the independence and of the complete existential theory of the postulates.

POSTULATE Q_1. K contains at least two distinct elements.

POSTULATE Q_2. If a, b are elements of $K, a\,|\,b$ is an element of K.

DEFINITION 1. $a' = a\,|\,a$.

POSTULATE Q_3. $a = (b\,|\,a)\,|\,(b'\,|\,a)$.

POSTULATE Q_4. $a\,|\,(b\,|\,c) = [(c'\,|\,a)\,|\,(b'\,|\,a)]'$.

3. *Sufficiency of the Postulates, Derivation of the Old Postulates.* It is seen that Postulates Q_1, Q_2, Q_3 are precisely my former postulates P_1, P_2, P_3, respectively. Hence, in order to establish the sufficiency of Postulates Q_1–Q_4 for Boolean algebras, it remains to derive from Q_1–Q_4 my former Postulate P_4, namely,

POSTULATE P_4. $a'\,|\,(b'\,|\,c) = [(b\,|\,a')\,|\,(c'\,|\,a')]'$.

This derivation I shall effect with the help of the following auxiliary theorems A and B.

THEOREM A. $a'' = a$, *where* $a'' = (a')'$.

THEOREM B. $a\,|\,b = b\,|\,a$.

The proofs of these theorems follow.

PROOF OF A. $a = (a\,|\,a)\,|\,(a'\,|\,a) = a'\,|\,(a'\,|\,a) = [(a'\,|\,a')\,|\,(a''\,|\,a')]' = a''$, by Q_3, Definition 1, Q_4, Q_3.

PROOF OF B. $a\,|\,b = a\,|\,b'' = a\,|\,(b'\,|\,b') = [(b''\,|\,a)\,|\,(b''\,|\,a)]' = [(b\,|\,a)\,|\,(b\,|\,a)]' = (b\,|\,a)'' = b\,|\,a$, by A, Definition 1, Q_4, A, Definition 1, A.

The proof of P_4 now follows.

PROOF OF P_4. $a'\,|\,(b'\,|\,c) = [(c'\,|\,a')\,|\,(b''\,|\,a')]' = [(c'\,|\,a')\,|\,(b\,|\,a')]' = [(b\,|\,a')\,|\,(c'\,|\,a')]'$, by Q_4, A, B.

Postulates Q_1–Q_4 are thus sufficient for Boolean algebras.

It can easily be verified that Postulates Q_1–Q_4 can be derived from Postulates P_1–P_4. Hence, Postulates P_1–P_4 and Postulates Q_1–Q_4 are, in fact, equivalent to one another.

I pass now to the derivation of Sheffer's postulates.

4. *Derivation of Sheffer's Postulates.* Sheffer's postulates are the propositions 1–5 below. In Postulates 3–5, the element a' is given by Definition 1 above, and in these postulates there is understood the condition *if the elements involved and their indicated combinations belong to K.*

POSTULATE 1. There are at least two distinct K-elements.

POSTULATE 2. Whenever a and b are K-elements, $a|b$ is a K-element.

POSTULATE 3. $(a')'=a$.

POSTULATE 4. $a|(b|b')=a'$.

POSTULATE 5. $[a|(b|c)]'=(b'|a)|(c'|a)$.

The derivation of these postulates from Q_1–Q_4 follow.

PROOF OF 1. By Q_1.

PROOF OF 2. By Q_2.

PROOF OF 3. By A.

PROOF OF 4. $a|(b|b')=[(b''|a)|(b'|a)]'=[(b|a)|(b'|a)]'=a'$, by Q_4, A, Q_3.

PROOF OF 5. $[a|(b|c)]'=[(c'|a)|(b'|a)]''=(c'|a)|(b'|a)$ $=(b'|a)|(c'|a)$, by Q_4, A, B.

I proceed, finally, to the derivation of the Whitehead-Huntington postulates.

5. *Derivation of the Whitehead-Huntington Postulates.* The Whitehead-Huntington postulates leave undefined a *class* K and two *binary operations* $+$ and $\times$, and are the propositions Ia, Ib, · · · , VI below. In Postulates IIIa–IVb is understood the condition *if the elements involved and their indicated combinations belong to K.* In V is understood the condition *if the elements z and u of* IIa *and* IIb *exist and are unique.*

POSTULATE Ia. $a+b$ is in K whenever a and b are in K.

POSTULATE Ib. ab is in K whenever a and b are in K.

POSTULATE IIa. There is an element z such that $a+z=a$ for every element a.

POSTULATE IIb. There is an element u such that $au=a$ for every element a.

POSTULATE IIIa. $a+b=b+a$.

POSTULATE IIIb. $ab=ba$.

POSTULATE IVa. $a+bc=(a+b)\ (a+c)$.

POSTULATE IVb. $a(b+c)=ab+ac$.

POSTULATE V. For every element a there is an element $\bar{a}$ such that $a+\bar{a}=u$ and $a\bar{a}=z$.

POSTULATE VI. There are at least two elements, a and b, in K such that $a\neq b$.

The proofs of Ia–VI follow.

DEFINITION 2. $a+b=(a|b)'$.

DEFINITION 3. $ab=a'|b'$.

PROOF OF Ia. By Definition 2, Definition 1, Q_2.

PROOF OF Ib. By Definition 3, Definition 1, Q_2.

PROOF OF IIa. The element $b'|b$, for any b, will serve as z. For, $a+(b'|b)=[a|(b'|b)]'=[(b'|a)|(b''|a)]''=a''=a$, by Definition 2, Q_4, Q_3, A.

PROOF OF IIb. The element $(b'|b)'$, for any b, will serve as u. For $a(b'|b)'=a'|(b'|b)''=a'|(b'|b)=[(b'|a')|(b''|a')]'=a''$ $=a$, by Definition 3, A, Q_4, Q_3, A.

PROOF OF IIIa. $a+b=(a|b)'=(b|a)'=b+a$, by Definition 2, B, Definition 2.

PROOF OF IIIb. $ab=a'|b'=b'|a'=ba$, by Definition 3, B, Definition 3.

PROOF OF IVa. $a+bc=[a|(bc)]'=[a|(b'|c')]'=(c''|a)$ $|(b''|a)]''=(c|a)|(b|a)=(a|b)|(a|c)=(a|b)''|(a|c)''=(a+b)'$ $|(a+c)'=(a+b)\ (a+c)$, by Definition 2, Definition 3, Q_4, A, B, A, Definition 2, Definition 3.

PROOF OF IVb. $a(b+c)=a'|(b+c)'=a'|(b|c)''=a'|(b|c)$ $=[(c'|a')|(b'|a')]'=[(a'|b')|(a'|c')]'=[(ab)|(ac)]'=ab+ac$, by Definition 3, Definition 2, A, Q_4, B, Definition 3, Definition 2.

PROOF OF V. The element a' will serve as $\bar{a}$. For, (1) $a'|a$ and $(a'|a)'$ may serve as z and u, respectively, by proof of IIa and proof of IIb; (2) $a+a'=(a|a')'=(a'|a)'$, by Definition 2, B; (3) $aa'=a'|a''=a'|a$, by Definition 3, A.

PROOF OF VI. By Q_1.

6. *The Complete Existential Theory of the Postulates*. The complete existential theory of postulates Q_1–Q_4 is given in the table below. In this table, a + sign in the ith place of the *character* of a system denotes that the system satisfies postulate Q_i, a − sign, that the system does not satisfy Q_i. In the concrete systems

of the table, the elements and the operations involved are all *arithmetic*. A symbol of the type $f(a, b)$ (mod p) in an arithmetic system denotes *the least positive integer* (*including* 0) *obtained from* $f(a, b)$ *by dropping multiples of* p. A blank for K and for $a|b$ indicates that there exists no system having the character concerned. The consistency of Q_1–Q_4 is given by system 1; the independence of Q_1, Q_2, Q_3, Q_4 is given, respectively, by systems 2, 3, 4, 5. The table follows.

System	Character	K	$a\|b$
1	$(++++)$	0, 1	$ab+1$ (mod 2)
2	$(-+++)$	0	0
3	$(+-++)$	0, 1	0/0
4	$(++-+)$	0, 1	0
5	$(+++-)$	0, 1	b
6	$(--++)$	0	0/0
7	$(-+-+)$	—	—
8	$(-++-)$	—	—
9	$(+--+)$	0, 1	$0/\{ab+a+1\}$ (mod 2)*
10	$(+-+-)$	0, 1, 2	$b+0/\{1-(a-a^2)(b-b^2)\}$ (mod 3)*
11	$(++--)$	0, 1	a
12	$(---+)$	—	—
13	$(--+-)$	—	—
14	$(-+--)$	—	—
15	$(+---)$	0, 1, 2	$a+0/\{1-(a-a^2)(b-b^2)\}$ (mod 3)*
16	$(----)$	—	—

* In system 9, Q_2 fails for $a=1$, $b=0$; Q_3 fails for $a=b=1$. In system 10, Q_2 fails for $a=b=2$; Q_4 fails for $a=b=0$, $c=1$. In system 15, Q_2 fails for $a=b=2$; Q_3 fails for $a=0$, $b=1$; Q_4 fails for $a=b=0$, $c=1$.

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