HANCOCK ON ALGEBRAIC NUMBERS

Foundations of the Theory of Algebraic Numbers. By H. Hancock. Vol. I. *Introduction to the General Theory.* 27+602 pp. Vol. II. *The General Theory.* 26+654 pp. New York, Macmillan, 1931–32.

Hancock's new book is easily the most voluminous treatise in the literature on algebraic numbers and ideal theory. The publication of such a handbook in English is in itself quite remarkable since ideal theory, both in algebraic fields and in abstract rings, mainly has been a continental or more specifically a German branch of mathematics. Textbooks in other languages have been few. Reid's book containing an elementary treatment of quadratic fields seems to have been the only contributions on the subject in English.

Since the time of Gauss the theory of algebraic numbers has been considered as one of the spires of mathematics and Hancock approaches his subject in an appropriate spirit: "By virtue of the simplicity of its foundations and the rigor of its deductions, arithmetic stands alone in the beauty and harmony of its truths. A divine gift, it offers proof that the mind is a reality attested by the sciences on the one hand, and the philosophies on the other. The province of arithmetic in this high position between science and philosophy, is both to serve and to be served in the quest of higher truths."

In the introduction the author discusses the natural problem of the foundations of ideal theory and decides to follow mainly the ideas of Dedekind, while their relations to the theories of Kronecker and Hensel are emphasized. This seems to be the simplest and most practical way of building up the ideal theory in algebraic fields. The relation between the various theories of Kummer, Dedekind, Zolotareff, Kronecker, Hensel, Prüfer, and others are fairly well known at the present time and a parallel development of all theories would be superfluous. Dedekind's point of view is in many ways both the simplest and the most general and this opinion, it seems to me, is confirmed by the facility with which the theory can be extended to abstract rings. It is true, however, that Hensel's theory has received a new impetus through the recent investigations of Krull on rings with generalized absolute values (Bewertungsringe).

Hancock's book deals exclusively with the classical ideal theory in algebraic fields. In the first volume one finds the foundations, beginning with the principal properties of polynomials, resultants, discriminants, and other preliminary notions. Then the definitions and main properties of fields, algebraic integers, units, bases, discriminants, and applications to quadratic and cubic fields are given, all following the well established lines of the theory. In the Chapters 5–7 one obtains a detailed discussion of the algebraic moduli introduced by Dedekind and in a form which is almost identical to the presentation in the supplements of the fourth edition of Dirichlet's *Number Theory.* The following Chapter 8 contains the elements of Kronecker's modular systems. In my opinion it would have been wiser to omit these chapters on modular theory in volume I and include them at the beginning of volume II where they are applied for the first time. Volume I would then have contained only the elementary parts of ideal theory. As it stands the modular theory forms an unnecessary obstacle to the ensuing chapters of the first volume.

The remaining half of the first volume deals with ideal theory in quadratic and cubic fields. It gives a comprehensive account of the elements of ideal theory and the clear and easy style should make it a good introduction for beginners in the subject. The presentation seems to be inspired by Sommer's book on algebraic numbers, since both the subject matter and the treatment show great similarities in both books. Chapter 9 treats the properties of ideals in quadratic fields and next follows a discussion of the quadratic law of reciprocity, the norm residue symbol of Hilbert, and the theorems on genera in quadratic fields. There is one chapter containing applications to Fermat's theorem for the exponents $n=3$ and $n=4$, another on the relation between quadratic forms and ideals, and a final chapter on cubic fields. The treatment of quadratic fields is a little cumbersome due to the necessity of splitting each proof into cases. This holds not only for Hancock's book but also for most other accounts. At the present time it should be possible to condense this theory by using the same methods as for general relative cyclic fields.

I should like to end this discussion of the first volume by an observation in connection with the determination of the prime ideal decomposition of a prime $p\neq 2$ in a quadratic field $K(m^{1/2})$. When $m\not\equiv 0 \pmod{p}$ the decomposition depends on the value of the symbol (m/p), that is, on the solvability of a congruence $x^2\equiv m \pmod{p}$. Both in Sommer and Hancock it is assumed that an eventual solution a is taken such that $a^2-m\not\equiv 0 \pmod{p^2}$ and then

$$p=\mathfrak{p}_1\mathfrak{p}_2=(p,\ m^{1/2}-a)(p,\ m^{1/2}+a).$$

It should be observed that this condition on a is unnecessary and that it could have been avoided by a slightly different proof.

The second volume contains a large number of topics from the theory of ideals. Hancock deduces the principal properties of ideals using the full apparatus of modular theory and following closely the presentation by Dedekind. This is certainly not the simplest method of introducing ideal theory, but on the other hand, it gives a more complete background for the results. However, I should have preferred to see the theory developed along the more modern lines of abstract ideal theory, giving, for instance, the much shorter proof by Krull for the fundamental theorem.

In the first two chapters the main properties of moduli, ideals, integral ideals, norms, and residue systems are discussed and then follows a chapter on prime ideals including the proof for the unique decomposition theorem, various elementary properties of prime ideals, and also an introduction of ideal classes. Another chapter gives an outline of Kronecker's ideal theory and then there is a second proof of the main theorem by Hurwitz. The theory of discriminant divisors is given in the same way as in Bachmann's book on the arithmetic of algebraic numbers. Dirichlet's theory of units is developed in two different ways and Minkowski's geometrical considerations are applied to the representation of the class number by means of the Dedekind zeta-function and to Dedekind's formula for the class number of rings. There is a short account of the theory of decomposable forms and next an extensive discussion of relative fields with applications to relative quadratic fields and with some elementary remarks on class fields. The results of Dedekind and Hilbert on the connection between ideals and group theory are derived, preceded by an account of the Galois theory of almost 100 pages. The final chapter contains some of the properties of p-adic numbers.

As a whole the second volume is not as well composed as the first. The various topics stand isolated with little or no attempt to connect them to an organic structure. The long elementary discussion of the Galois theory might well have been omitted since a book on algebraic numbers cannot be expected to be a textbook on all algebraic theories. Some of the topics are rather obsolete: as an example I mention the theory of common index divisors which, according to more recent points of view, owe their importance to the imperfection of the method applied and not to their intrinsic properties.

I feel certain that Hancock's book will be used as textbook in a large number of courses on algebraic numbers and that it will prove very helpful, particularly for those students whose knowledge of the German language does not permit them to read the sources with facility. As I have already mentioned, the book is clearly written and several of my graduate students in a course on ideal theory have expressed their satisfaction with its presentation.

However, it cannot be denied that the book has various defects, some of which I consider to be serious. First of all, it is too bulky. I have already indicated several chapters which could have been omitted without great loss But there are other reasons for the great size of the book than the richness of the material: almost every theorem of importance is proved in at least two ways and several in still more ways. The fundamental theorem on the unique decomposition of ideals is proved three or four times in Volume 2 and in addition the full ideal theory is developed independently both for quadratic and cubic fields in Volume 1. In some cases it may clarify a theory to give different proofs, but to this extent it seems an unnecessary repetition only tiring the reader.

The main sources for Hancock's book are obviously: Sommer's *Lectures on Number Theory*; Bachmann's *General Arithmetic of Number Fields*; Dedekind's Supplements to Dirichlet's *Number Theory*; Hilbert's *Report*; and Weber's *Algebra*. These are all admirable contributions to the theory of ideals, but it should be remembered that they were published more than 25 years ago and most of them are considerably older. Since then the theory of algebraic numbers and ideals has made great strides, with new progress in older problems and new channels opened up. Of this development Hancock's book gives no indication. It leaves the student without bringing him in touch with modern problems and without bridging the gap to present day periodical literature.

Hancock's book will probably be widely used as a reference book and hence have an influence on future English nomenclature. This prompts me to make a final observation about the notation used in the book. It seems to me that the translations of the German terms are not always fortunate, although I admit that it may be a matter of taste. I should have preferred the term field to realm, fundamental field to stock realm, discriminant to basal invariant, conductor to ring leader, etc. Terms like Gattung and genuses should have been avoided.

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