NOTE ON A SPECIAL CYCLIC SYSTEM*

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1. *Introduction.* This note is concerned with a special cyclic system.† Let S be a surface referred to any orthogonal system, and T the trihedral whose x-axis is tangent to the curve $v=\text{const.}$ The equations

$$(1)\qquad x=R(1+\cos\theta),\quad y=0,\quad z=R\sin\theta,$$

define a two-parameter family of circles C normal to S; and the necessary and sufficient conditions that C shall constitute a cyclic system are

$$(2)\ \xi\frac{\partial R}{\partial v}+R\eta_1 r=0,\quad R(pr_1-p_1r)-q_1\left(\xi+\frac{\partial R}{\partial u}\right)+q\frac{\partial R}{\partial v}=0.$$

It is readily seen that the first of equations (2) may be written‡

$$\xi\frac{\partial R}{\partial v}-R\frac{\partial \xi}{\partial z}=0;$$

hence

$$(3)\qquad R=U\xi,$$

where U is a function of u alone. Using (3) we may write the second equation of (2) in the form

$$(4)\quad U\xi(pr_1-p_1r)-q_1\left(\xi+\xi U'+U\frac{\partial\xi}{\partial u}\right)-qU\eta_1 r=0.$$

We shall therefore replace equations (2) by (3) and (4).

2. *The Inversion of C.* If we invert the circles C relative to the circles $x^2+z^2=K^2$, $y=0$, where K is any constant, we get the following system of lines L,

$$(5)\qquad x=\frac{K^2}{2R},\quad y=0,$$

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† See Eisenhart, *Differential Geometry of Curves and Surfaces*, Ex. 11, p. 444.

‡ Eisenhart, p. 170.

which will constitute a rectilinear congruence. Let us determine the condition that this congruence be normal.

Any point on the lines (5) will have the coordinates $(K^2/(2R), 0, z)$. The necessary and sufficient condition that (5) shall define a normal congruence is that $\delta z = 0$ for all values of dv/du. Hence

$$\frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv - \frac{K^2}{2R}(q du + q_1 dv) \equiv 0,$$

or

$$\frac{\partial z}{\partial u} = \frac{K^2 q}{2R}, \quad \frac{\partial z}{\partial v} = \frac{K^2 q_1}{2R}.$$

The condition of integrability is that $\partial^2 z/\partial u \partial v = \partial^2 z/\partial v \partial u$; hence

$$R\frac{\partial q}{\partial v} - q\frac{\partial R}{\partial v} = R\frac{\partial q_1}{\partial u} - q_1\frac{\partial R}{\partial u},$$

which by means of (3) becomes*

$$U\xi(rp_1 - r_1 p) + qU\,\eta_1 r + q_1\left(\xi U' + U\frac{\partial \xi}{\partial u}\right) = 0.$$

On making use of (4), this further reduces to $q_1 = 0$; hence we have the following theorem.

THEOREM 1. *A necessary and sufficient condition that the lines L, obtained by the above inversion of the cyclic system C, shall form a normal congruence, is that S be referred to its lines of curvature.*

We readily find that for $q_1 = p = 0$, (4) becomes $r(\xi p_1 + \eta_1 q) = 0$. Two cases are to be considered.

(a) If $r = 0$, the curves $v = \text{const.}$ are geodesics; and since they are lines of curvature, they are plane curves. Consequently the planes of C constitute but a one-parameter family, and their envelope is the developable surface which is readily seen to be one of the focal sheets of S. We note also from (2) that for $r = 0$, R is a function of u alone.

(b) If $\xi p_1 + \eta_1 q = 0$, it is readily seen† that $D:E = D'':G$. Hence S is in this case either a sphere or a plane. If S is a sphere the planes of C pass through the center.‡

* Eisenhart, p. 168.

† Eisenhart, p. 174.

‡ Eisenhart, p. 441.

3. *The Focal Points of C.* The direction cosines of an arbitrary tangent of C are $(-\sin\theta, 0, \cos\theta)$. Since the displacements $\delta x, \delta y, \delta z$ of the focal points of C must be in the direction of the tangents we have $\cos\theta\,\delta x+\sin\theta\,\delta z=0$, $\delta y=0$. On using (1), these equations may be written

$$
(6)\qquad \begin{aligned}
&\frac{x}{R}\frac{\partial R}{\partial u}du+\frac{x}{R}\frac{\partial R}{\partial v}dv+\frac{\xi(x-R)}{R}du-z(q\,du+q_1dv)=0,\\
&\eta_1dv+x(r\,du+r_1dv)-z(p\,du+p_1dv)=0.
\end{aligned}
$$

Obviously, we must have

$$
(7)\qquad \begin{vmatrix}
\frac{x}{R}\frac{\partial R}{\partial u}+\frac{\xi(x-R)}{R}-qz & \frac{x}{R}\frac{\partial R}{\partial v}-q_1z\\
rx-pz & \eta_1+r_1x-p_1z
\end{vmatrix}=0.
$$

Hence the focal points of C, which are at most four in number, are given by (1) and (7).

Let us now consider the case when the inversion of C leads to a normal rectilinear congruence. We have seen that we must have $p=q_1=0$, and $r(\xi p_1+\eta_1 q)=0$. The first case, (a), is of some interest; for $p=q_1=r=0$, equation (7) becomes*

$$
(8)\qquad \left(\frac{\partial R}{\partial u}+\xi\right)x-Rqz-R\xi=0,\qquad r_1x-p_1z+\eta_1=0.
$$

We note that the two centers of principal curvature, $(0, 0, -\xi/q)$ and $(0,0, \eta_1/p_1)$, lie in the planes (8), while the second plane of (8) also contains the center of geodesic curvature, $(-\eta_1/r_1, 0, 0)$, of the curve $u=\text{const}$. Hence we have the following theorem.

THEOREM 2. *If the above inversion of C leads to a normal rectilinear congruence, and the curves $v=$const. are geodesics, $(r=0)$, then two of the focal points of C are collinear with the center of principal curvature of the curve $v=$const., and the other two are collinear with the centers of principal and geodesic curvature of the curve $u=$const.*

From (6) it is evident that for $p=q_1=r=0$, and $R=f(u)$, du and dv are factors of these equations. Hence we have the following theorem.

* R is a function for u alone for (a).

THEOREM 3. *If the above inversion of C leads to a normal rectilinear congruence, and the curves $v = const.$ are geodesics, then the circles of C which have an envelope are those which correspond to the lines of curvature on S.*

4. *The Focal Points of L. The Developables of L.* As the vertex of T is displaced along the curves of S which define the developables of L, the displacements δx and δy of the focal points will be zero. Hence from (5), when L is a normal congruence, we have

$$(9)\qquad \begin{aligned} &\frac{K^2}{2R^2}\left(\frac{\partial R}{\partial u}du + \frac{\partial R}{\partial v}dv\right) - \xi du - zq du = 0, \\ &\eta_1 dv + \frac{K^2}{2R}(r du + r_1 dv) - z p_1 dv = 0. \end{aligned}$$

The elimination of dv/du between equations (9) gives us a quadratic in z whose roots determine the focal points of L; the elimination of z gives the equation of the curves on S defining the developables of L.

For (a), that is for $r = 0$, $R = f(u)$, the focal points of L as determined by (9) are

$$(10)\qquad z_1 = \frac{1}{q}\left(\frac{K^2}{2R^2}\,\frac{\partial R}{\partial u} - \xi\right), \qquad z_2 = \frac{1}{p_1}\left(\frac{K^2 r_1}{2R} + \eta_1\right).$$

When we put (5) in the second of equations (8) and solve for z we get the value of z_2 in (10). Hence one of the focal points of L is collinear with the two focal points of C determined by the second member of (8). It is readily shown from (5), (8), and (10) that the other focal point of L is not collinear with the other two focal points of C.

From (9) it is readily seen that for (a), du and dv are factors of these equations. Hence we have the following theorem.

THEOREM 4. *If the above inversion of C leads to a normal rectilinear congruence L, and the curves $v = const.$ are geodesics, the lines of curvature on S define both the developables of L, and those families of C which have an envelope.*

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