principle for points on an algebraic curve, and the 28 double tangents of a quartic, and closes with five pages on the Riemann-Weierstrass theta-theorem.

There is little new in this book beyond the occasional use of the plane of function theory, nor is it encyclopedic. Although much of the language suggests geometry, mention is made only of those geometrical concepts which are strictly essential to the topic. The domain of variation is sometimes viewed as a curvilinear polygon, sometimes as a dissected Riemann surface, sometimes as a plane curve, sometimes as a curve in n-space. It is never the mere abstract bearer of algebraic groups of points, as handled successfully by the modern Italians. The pervading points of view are those of Klein with wordiness excised and with more explicit proofs. Charm and thrill are sometimes found in proceeding from the simple to the advanced, from the concrete to the abstract, in the developing conviction of power to handle problems, in catching glimpses of domains yet awaiting the adventurer. These qualities often apparent in the original papers of the investigators, and retained in such expositions as that of Appell and Goursat, are here consciously sacrificed to an impersonal economy in abstract and logical treatment which may be justified if this interesting subject is to be compressed to fit into a general treatise on function theory.

A. A. Bennett

VEBLEN-WHITEHEAD—FOUNDATIONS OF DIFFERENTIAL GEOMETRY

The Foundations of Differential Geometry. By Oswald Veblen and J. H. C. Whitehead. (Cambridge Tracts in Mathematics and Mathematical Physics, No. 29.) Cambridge, at The University Press, 1932. New York, The Macmillan Company. ix+96 pp.

In the preface to his earlier work on *Invariants of Quadratic Differential Forms* (Cambridge, 1927), Professor Veblen regretfully stated that even a short discussion of differential geometry had been crowded out because of the limited space available. If the necessity for this omission led to the writing of the present tract, it was a fortunate circumstance, for the result is a profoundly stimulating book.

The title, *The Foundations of Differential Geometry,* is scarcely broad enough to describe the contents, for the book is a critical study of the foundations of all mathematical systems which have been called geometries. The authors have kept strictly to their theme, namely the logical foundations of geometry, and have not yielded to the temptation to develop any particular geometry much beyond the postulates by which it is defined. While the treatment is philosophical and highly critical, yet it is so skillfully done that the book is easy to read—a fact which testifies to the genius of the authors for mathematics of this kind.

A rigid definition of geometry is not attempted, on the ground that any objective definition of geometry should include the whole of mathematics. The question is dismissed with the statement that a branch of mathematics is called a geometry because the name seems good, on emotional and traditional grounds, to a sufficient number of competent people. It is well known that in recent years geometry has outgrown the limitations placed upon it by Klein in his Erlanger Programm of 1870. Without attempting to be dogmatic, the authors give systems of axioms which they believe provide an adequate foundation for any of the differential geometries which are now being studied.

First the arithmetic space of n dimensions is discussed. An arithmetic point is merely an ordered set of n real numbers. The set of points linearly dependent on $k+1$ linearly independent points constitutes an arithmetic linear k-space. The usual theorems on systems of linear homogeneous equations are developed in geometric terminology.

The translation group, the centered affine group, the affine group, the orthogonal group, and the euclidean metric group are defined when operating upon the points of an arithmetic n-space, and the corresponding geometries are defined according to the idea of Klein.

Another specification of a geometry, namely as a *coordinate geometry*, is now discussed. A coordinate system is defined as a correspondence $P \rightarrow x$ between a set $[P]$ of points (undefined elements of the geometry) and a set of arithmetic points $[x]$. Another set of undefined elements are termed *preferred coordinate systems*. Four axioms relating these undefined elements and the group G of transformations of arithmetic points into arithmetic points establish the geometry of the group G.

The axiomatic approach to a differential geometry is less simple. An important concept introduced in this connection is that of *pseudo-group*. A set of transformations constitute a pseudo-group if (i) the resultant of two transformations is in the set provided it exists, and if (ii) the set contains the inverse of each transformation of the set. The word *geometric object* is preferred by the authors to the term *invariant* when invariance under a pseudo-group is intended. A geometric object in a simple manifold determines a structure and therefore a space. The geometry of this space is the geometry of the object. Geometric objects (and the spaces which they determine) are classified by means of the pseudo-group of regular point transformations. This classification of geometric objects is a natural generalization of Klein's point of view, a geometry defined by a geometric object falling within the categories of the Erlanger Programm if and only if the geometric object is characterized by its group of automorphisms.

A function of n variables, defined for all points of a region X, is of class u if it and its derivatives of orders $\leqq u$ exist and are continuous at each point of X. A transformation $y^i = y^i(x)$ is of class u if each of the n functions $y^i(x)$ is of class u. An allowable coordinate system of class u in a simple manifold of class $u' \geqq u$ is defined by the conditions:

1. If $[P]$ is the image in a preferred coordinate system K of an arithmetic region $[x]$, the correspondence $P \rightarrow x$ determined by K is an allowable coordinate system.

2. If $[x]$ is an arithmetic region, $P \rightarrow x$ an allowable coordinate system, and $x \rightarrow y$ a regular transformation of class u, then the resultant transformation $P \rightarrow y$ is an allowable coordinate system.

Chapter IV is concerned with the theory of k-cells in n-space and the local structure or infinitesimal geometry at a point P. Chapter V is on tangent spaces in general, tangent and osculating Riemannian spaces, affine connections, and parallel and affine displacements. This chapter contains a very interesting critical examination of the foundations of tensor analysis.

Chapter VI, entitled *A Set of Axioms for Differential Geometry*, is the heart of the book. The axioms are stated in terms of *points* and *allowable coordinate systems*, points being undefined, and the allowable coordinate systems constituting an undefined class of biunique correspondences $P \to x$ between sets of points and sets of arithmetic points in the arithmetic n-space.

The first group of axioms describe the local structure completely:

A_1. The transformation of coordinates between two allowable coordinate systems which have the same domain is regular, provided one of them, at least, has a region for its arithmetic domain.

A_2. Any coordinate system obtained by a regular transformation of coordinates from an allowable coordinate system is allowable.

A_3. The correspondence in which each point of an n-cell corresponds to its image in an allowable coordinate system is an allowable coordinate system.

Let $[P]$ and $[Q]$ be the domains of biunique coordinate systems $P \to x$ and $Q \to y$ respectively, $[x]$ and $[y]$ their arithmetic domains, and $[R]$ the intersection of $[P]$ and $[Q]$. If the correspondences are consistent for all points of $[R]$, they define together a coordinate system called their *union*.

The second group of axioms characterizes the class of allowable coordinate systems in a space satisfying the axioms A:

B_1. Any coordinate system which is the union of a set of allowable coordinate systems whose domains are n-cells, is allowable.

B_2. Each allowable coordinate system is the union of a set of allowable coordinate systems whose domains are n-cells.

The third group of axioms are of a topological nature:

C_1. If two n-cells have a point in common, they have in common an n-cell containing this point.

C_2. If P and Q are any two distinct points, there is an n-cell C_P containing P, and an n-cell C_Q containing Q, such that C_Q has no point in common with C_P.

C_3. There exist at least two points.

A surprising number (seventeen) of non-trivial theorems are deduced from these seemingly mild axioms, as, for instance, the result that the totality of regular point transformations between regions in a regular manifold is a pseudo-group.

However, it is noted that the axioms describe a large class of spaces which are not all equivalent, so that further special postulates must be added to secure special geometries. Some of these are considered in the last chapter.

As Mr. Mencken would conclude, "There is an index."

C. C. MacDuffee