SPACES ADMITTING COMPLETE ABSOLUTE PARALLELISM*

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At the Colloquium of this Society held at Ithaca in September, 1925, I set forth, under the title *The New Differential Geometry*, certain developments which had taken place during the preceding ten years, growing out of the concept of infinitesimal parallelism for Riemannian spaces proposed by Levi-Civita. When these lectures were published in book form [1]† in 1927, the book included also material which had been developed in the interim. Since that time there have been many further developments. Instead of trying to make a full survey of these, I have decided to limit my paper to the theory of linearly connected manifolds admitting a complete absolute parallelism.

Levi-Civita [2] introduced the concept of parallelism in a Riemannian space as a means of giving an invariantive interpretation to the curvature of the space. Since a Riemannian space of n dimensions, V_n, may be thought of as a sub-space of a euclidean space of suitable dimensions, Levi-Civita applied the concept of parallelism of the euclidean space to vectors tangential to the sub-space. In fact, vectors a and a' at two nearby points P and P' were defined to be parallel, if the angles between a and a tangent vector f at P and a' and f are equal from the euclidean point of view for every tangential vector f. Analytically this leads to the result that, if in terms of general coordinates x^i in V_n the coordinates of P and P' are x^i and x^i+dx^i, then ξ^i and $\xi^i+d\xi^i$ are the components of parallel directions at P and P', provided

$$(1) \qquad d\xi^i + \left\{ {i \atop jk} \right\} \xi^j dx^k = 0, \qquad (i, j, k = 1, \cdots, n),$$

where $\left\{ {i \atop jk} \right\}$ are the Christoffel symbols of the second kind

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† Such references are to the items in the bibliography at the end of this article.

formed with respect to the fundamental tensor g_{ij} of V_n.* Although the metrical properties of the enveloping euclidean space were used in the geometrical definition of parallelism, the analytical result involves only the quantities of V_n, that is, it is intrinsic, a result which Levi-Civita considered necessary for an appropriate definition.

Let P and Q be two nearby points of a geodesic C. Through them geodesics are drawn in parallel directions, thus making the same angle ψ with C, and on the geodesics equal lengths are laid off from P and Q with end points P' and Q', so that we have a parallelogrammoide $PQP'Q'$. If $P'Q'$ and PQ denote the lengths of these respective sides of $PQP'Q'$, then as Levi-Civita has shown [**2**, p. 191] the Riemannian curvature at P for the orientation determined by PQ and PP' is equal to the ratio of $(P'Q')^2-(PQ)^2$ and the square of the area of $PQP'Q'$.

Recognizing that parallelism is an affine property and thus is not limited to spaces with an assigned metric, Weyl [**3**] generalized equations (1) in the form

$$d\xi^i + \Gamma^i_{jk}\xi^j dx^k = 0, \tag{2}$$

where the Γ's are functions of the x's. Consider three nearby points P, P_1, and P_2 of coordinates x^i, $x^i+d_1x^i$, and $x^i+d_2x^i$. The coordinates of the end point of the vector at P_2 parallel to d_1x^i at P are

$$x^i + d_2x^i + d_1x^i + d_2d_1x^i + \Gamma^i_{jk}\, d_1x^j d_2x^k.$$

Interchanging the subscripts 1 and 2, we have the coordinates of the end point of the vector at P_1 parallel to d_2x^i at P. These points coincide when and only when

$$\Gamma^i_{jk} = \Gamma^i_{kj}, \tag{3}$$

as in the case of the Christoffel symbols. Weyl imposed this condition in his definition of infinitesimal parallelism, and of an affinely connected manifold. The quantities Γ^i_{jk} are called the *coefficients of the affine connection*.

When the coordinates x^i of the space undergo a non-singular

* Throughout this paper we use the convention that a term containing a repeated index indicates the sum of such terms as the index takes on all possible values.

transformation into coordinates x'^i, and we denote by Γ'^i_{jk} the coefficients of the connection in the x''s, we have

$$\frac{\partial^2 x^i}{\partial x'^\alpha \partial x'^\beta} + \Gamma^i_{jk} \frac{\partial x^j}{\partial x'^\alpha} \frac{\partial x^k}{\partial x'^\beta} = \Gamma'^\gamma_{\alpha\beta} \frac{\partial x^i}{\partial x'^\gamma} \cdot \tag{4}$$

From these equations it follows that the coordinates x'^i can be chosen in many ways so that at a given point $\Gamma'^\gamma_{\alpha\beta} = 0$, and consequently it follows from (2) that parallel vectors at P and nearby points have the same components. Consequently, in the neighborhood of a point the situation is the same as that which obtains throughout the whole of euclidean space in terms of cartesian coordinates. Moreover, this property is true only in case equations (3) are satisfied, and thus it may be taken as a characteristic property of Weyl's affine connection.

Veblen and I, [4] and [1], developed this theory by taking as basis the paths, that is, the integral curves, of the system of equations

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0.$$

These curves are the straightest lines for the definition (2) of parallelism, in that the tangents to any path are parallel with respect to the path. From this basis one develops naturally a projective geometry as well as an affine one.

A necessary and sufficient condition that there exists a coordinate system for which $\Gamma'^\gamma_{\alpha\beta} = 0$ at every point, in which case we have a euclidean space and the coordinates are cartesian, is that

$$B^h_{ijk} = 0, \tag{5}$$

where

$$B^h_{ijk} = \frac{\partial \Gamma^h_{ik}}{\partial x^j} - \frac{\partial \Gamma^h_{ij}}{\partial x^k} + \Gamma^l_{ik} \Gamma^h_{lj} - \Gamma^l_{ij} \Gamma^h_{lk}. \tag{6}$$

If in the equations

$$\frac{d\xi^i}{dt} + \Gamma^i_{jk} \xi^j \frac{dx^k}{dt} = 0 \tag{7}$$

we put

(8) $$x^i = \phi^i(t),$$

a solution of the system (7) is determined by the initial values of the ξ's. Thus at each point of the curve (8) we have a vector and all of these vectors are said to be parallel. Consequently, we have distant parallelism of vectors, but in general, as follows from (7), the vector at a point Q parallel to a vector at a point P depends upon the curve joining P and Q. Thus the parallelism is not absolute as in the case of euclidean space. Levi-Civita introduced this idea, using (7) in which the Γ's are the Christoffel symbols, and Weyl generalized it to affine connections.

If parallelism is to be independent of the curve, ξ^i must satisfy the equations

(9) $$\frac{\partial \xi^i}{\partial x^k} + \Gamma^i_{jk}\xi^j = 0, \quad (i, j, k = 1, \cdots, n).$$

Each set of solutions of these equations defines a field of absolutely parallel vectors. In order that there may be n independent fields, it is necessary that the conditions of integrability of the above equations, that is, $\xi^i B^h_{ijk} = 0$, be satisfied identically. Consequently, equations (5) must hold and the space be euclidean.

We have remarked that Weyl imposed the condition that the functions Γ^i_{jk} be symmetric in j and k in order to insure that in the neighborhood of a point parallelism be euclidean. If this restriction is not made, we say that the affine connection is *asymmetric* and take as the coefficients L^i_{jk}, denoting by Γ^i_{jk} and Ω^i_{jk} the symmetric and asymmetric parts, so that

(10) $$L^i_{jk} = \Gamma^i_{jk} + \Omega^i_{jk}.$$

In place of (4) we have

(11) $$\frac{\partial^2 x^i}{\partial x'^\alpha \partial x'^\beta} + L^i_{jk}\frac{\partial x^j}{\partial x'^\alpha}\frac{\partial x^k}{\partial x'^\beta} = L'^\gamma_{\alpha\beta}\frac{\partial x^i}{\partial x'^\gamma},$$

from which follow (4) and

(12) $$\Omega^i_{jk}\frac{\partial x^j}{\partial x'^\alpha}\frac{\partial x^k}{\partial x'^\beta}\frac{\partial x'^\gamma}{\partial x^i} = \Omega'^\gamma_{\alpha\beta},$$

so that the Ω's are components of a tensor. Any set of functions L^i_{jk} and $L'^\gamma_{\alpha\beta}$ satisfying (11) are said to determine an asymmetric affine connection. If in (9) we replace Γ^i_{jk} by L^i_{jk}, we have the equations of condition that a field of vectors be absolutely parallel and their condition of integrability is $\xi^i L^h_{ijk}=0$, where L^h_{ijk} is the same expression in the L's as B^h_{ijk} (6) is in the Γ's. Consequently a necessary and sufficient condition that there be n independent fields of parallel vectors is that

$$L^h_{ijk} = 0;$$

in this case we say that the space admits *complete absolute parallelism*, and that it is of *zero curvature*. Since Ω^i_{jk} are components of a tensor, if they are zero in one coordinate system at a point they are zero in every system. Hence when, and only when, the connection is symmetric is it possible for all the coefficients of the connection to be zero at a point. Consequently spaces with asymmetric connections of zero curvature are not euclidean, but pseudo-euclidean in that they admit complete absolute parallelism.

If in the case of a space of zero curvature we denote by λ_α the components of n independent fields of parallel vectors, i indicating the component and α the vector, we have

$$\frac{\partial \lambda^i_\alpha}{\partial x^k} + \lambda^j_\alpha L^i_{jk} = 0. \tag{13}$$

Since the matrix $\|\lambda^i_\alpha\|$ is of rank n, quantities λ^α_i are defined uniquely by

$$\lambda^i_\alpha \lambda^\alpha_j = \delta^i_j, \quad \lambda^\beta_\alpha \lambda^\beta_i = \delta^\beta_\alpha, \tag{14}$$

where a δ is equal to one or zero, according as its indices are the same or different. By means of these functions we have from (13)

$$L^i_{jk} = -\lambda^\alpha_j \frac{\partial \lambda^i_\alpha}{\partial x^k} = \lambda^i_\alpha \frac{\partial \lambda^\alpha_j}{\partial x^k}. \tag{15}$$

Conversely, if we have n independent vectors λ^i_α and define L^i_{jk} by (15) with the aid of λ^α_i, the quantities so determined in any two coordinate systems satisfy (11), and thus determine an affine connection with respect to which the vectors λ^i_α are absolutely parallel. In 1922–1923 Weitzenböck [**5**, p. 319] showed that the functions (15) satisfy (11) and used them as a basis for covariant differentiation, but made no reference to the parallelism which they define. However, in 1923 Vitalli [6] used them to define parallelism, and pointed out that each field of vectors is absolutely parallel.

In 1917 Hessenberg [7] developed a differential geometry based upon an ennuple of independent vectors and developed a type of differentiation based thereon. Two years later König [8] analyzed these concepts in the light of Weyl's contribution and called attention to the fact that the affine connection defined thereby is not symmetric. So far as I know this is the first reference to the use of an asymmetric connection, although evidently it was considered by Weyl and not used for the reasons previously mentioned. In 1922–1923 Cartan investigated the geometry of a space with an affine connection based on an ennuple of vectors and developed the concept of equipollance involving the idea of parallelism. He showed that the vanishing of the curvature tensor is the condition for complete absolute parallelism [9, p. 368] and thus he seems to have been the first writer to deal explicitly with this concept. Schouten [10] made an extensive study of types of linear connection.

If we put

$$g_{ij} = \lambda^{\alpha}_{i}\lambda^{\alpha}_{j}, \qquad g^{ij} = \lambda^{i}_{\alpha}\lambda^{j}_{\alpha}, \tag{16}$$

we have

$$g_{ij}g^{jk} = \delta^{k}_{i}.$$

The tensor g_{ij} thus defined may be taken as the basis for a Riemannian metric in the affinely connected manifold, in which case the vectors λ^{i}_{α} form an orthogonal ennuple. The Christoffel symbols formed with respect to g_{ij} are in the following relation to the coefficients of the affine connection:

$$\left\{ {i \atop jk} \right\} = \Gamma^{i}_{jk} + g^{ih}(g_{jl}\Omega^{l}_{hk} + g_{kl}\Omega^{l}_{hj}). \tag{17}$$

Vitalli [6] proposed this introduction of a metric.

What I have presented thus far was a matter of record, when in 1928 Einstein [11] proposed a unified theory of gravitation and electricity based upon the concept of a Riemannian space admitting distant parallelism. He was unaware of the existence of the requisite mathematical knowledge and developed it anew. He said "The new unitary field theory is based on the following mathematical discovery: There are continua with a Riemannian metric and distant parallelism which nevertheless are not euclidean." Later he gave up hope of founding a satisfactory theory on such a basis and recently turned to what is essentially a generalized projective geometry. The elements of such a geometry were presented in my colloquium lectures, and I might have chosen this subject for my address. But this theory and its application to relativity are presented by Veblen in his forth-coming book, *Projektive Relativitätstheorie*, one of Springer's "Ergebnisse der Mathematik."

Bianchi, [**12**, p. 517], showed that a simply transitive group is the group of motions of determinate Riemannian manifolds. In 1925 I [**13**] gave a new proof of this result, making use of the linear connection whose coefficients are defined by

$$L^{i}_{jk} = - \xi^{\alpha}_{k}\frac{\partial \xi^{i}_{\alpha}}{\partial x^{j}} = \xi^{i}_{\alpha}\frac{\partial \xi^{\alpha}_{k}}{\partial x^{j}}, \tag{18}$$

where ξ^i_α are the vectors of the transitive group and ξ^α_i are given by equations of the form (14), and showed that the connection of coefficients L^i_{jk} has zero curvature. Consequently the equations

$$\frac{\partial g_{ij}}{\partial x^{k}} = g_{il}L^{l}_{jk} + g_{jl}L^{l}_{ik} \tag{19}$$

are completely integrable, and it can be shown that any solution is the fundamental tensor of a Riemannian space admitting the group as a group of motions. Since the connection is of zero curvature, there are n independent fields of absolutely parallel vectors, say λ^i_α, in terms of which L^i_{jk} are expressed in the form (15). The quantities g_{ij} given by (16) satisfy (19), and each solution of (19) is expressible in terms of suitable linear combinations of the λ's with constant coefficients. We remark furthermore that the λ's are the vectors of the simply transitive group reciprocal to the given group. They may be used as in (18) to determine a second affine connection of zero curvature, whose coefficients $\bar{L}^i_{jk}$ satisfy the condition $\bar{L}^i_{jk} = \bar{L}^i_{kj}$.

When an affine connection has zero curvature, an ennuple of absolutely parallel vectors are the vectors of a simply transitive group, when and only when the first covariant derivatives of Ω^i_{jk} with respect to the L's are zero. In this case Γ^i_{jk} and Ω^i_{jk} satisfy the conditions

$$B^h_{ijk} = \Omega^l_{jk}\Omega^h_{il}.$$

In accordance with the theory of Lie, if

$$x'^i = f^i(x^1, \cdots, x^n; a^1, \cdots, a^r)$$

are the equations of an r-parameter continuous group, then

$$\frac{\partial x'^i}{\partial a^t} = \xi^i_\alpha(x')A^\alpha_t(a), \qquad \begin{pmatrix} i = 1, \cdots, n; \\ \alpha, t = 1, \cdots, r \end{pmatrix},$$

where ξ^i_α are the vectors of the group, and under a non-singular transformation of the a's the A's are related as covariant vectors. They may be interpreted as vectors in the group-space S of coordinates a^t. By means of equations similar to (14) we obtain an ennuple of contravariant vectors A^t_α, and as is well known they are the vectors of a simply transitive group. Consequently they, and the vectors of the reciprocal group, determine two affine connections of zero curvature for the group space. This theory has been studied extensively by Cartan [**14, 15**] and Schouten [**14, 16**] as setting forth the geometry of the two-parameter groups of a given group. In particular the trajectories of a one-parameter sub-group of either parameter group are parallel paths of S.

Einstein and Mayer [**17**] in developing the unified theory on the basis of absolute parallelism considered the rotation group of motions of the manifold in order to determine a spherically-symmetric space-time. Robertson [**18**] considered the general question of motions of a linearly connected manifold of zero curvature. His definition is equivalent to saying that if equations (11), in which L'^i_{jk} are the same functions of the x''s as the corresponding L^i_{jk} are of the x's, admit a solution involving one or more parameters, these solutions define a motion. When a metric is assigned, as in (16), the equations of motion imply the Killing equations. In this case a group of maximum order, namely $n(n+1)/2$, is realized only in case $\Omega^i_{jk}=0$ and the space is euclidean, except when $n=1$ or 3. In the latter case the space is of constant curvature. If the curvature is positive and the metric is positive definite, the two parallelisms are those discovered long ago by Clifford. Cartan [**19**, p. 704] had shown previously that the two types of parallelism of Clifford could be identified with parallelism in a linearly connected manifold of zero curvature.

The differential invariants of a linearly connected manifold of zero curvature have been studied by Griss [**20**], Weitzenböck [**21**], and Bortolotti [**22**], but we shall not set forth their results.

In presenting this review of the development of the concept of complete absolute parallelism I may quite unintentionally have omitted references to writers which should be included. Furthermore I have not attempted to give a complete bibliography of the subject; the reader will find further references in the papers listed, and in particular in [**21**], [**23**], and [**24**].

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