ON THE EXISTENCE OF THE ABSOLUTE MINIMUM IN PROBLEMS OF LAGRANGE†

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In a previous paper,‡ I remarked that the theorems obtained are applicable to problems of Lagrange in which the side conditions are of the form

$$(1) \qquad g_\mu(x, y_1, \cdots, y_k) = 0, \qquad (\mu = 1, \cdots, m).$$

It is easily seen also that *the proofs* there made are applicable under suitably *weakened hypotheses* (to be stated below) to problems of Lagrange in which appear side conditions which are *differential equations* of the form

$$(2) \qquad y_\sigma' = h_\sigma(x, y_1, \cdots, y_k), \quad (\sigma = 1, \cdots, s).$$

Calculus of variations problems in which the integrand involves higher derivatives are reducible to Lagrange problems of this type, as are also isoperimetric problems in which the integrands of the integrals to be kept constant depend only on the coordinates. Examples in which the stronger hypotheses of the paper cited do not hold are given in the final paragraph.

We suppose that the integrand function $f(x, y_1, \cdots, y_k, y_1', \cdots, y_k')$ and the functions g_μ, h_σ, together with the partial derivatives $f_{y_i'}$, $g_{\mu x}$, $g_{\mu y_i}$, are defined and continuous for all points (x, y) in a closed domain A and for all y'. Let R^* denote the set of all points (x, y, y') having (x, y) in A and satisfying equations (1), (2) and

$$(3) \qquad g_{\mu x} + g_{\mu y_i} y_i' = 0, \qquad (\mu = 1, \cdots, m).$$

An *admissible curve* C, $y_i = y_i(x)$, is one which is absolutely continuous and has all its elements (x, y, y') in R^*.§ Then if K is a closed class of absolutely continuous curves, the sub-class K^*

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‡ *On the existence of the absolute minimum in space problems of the calculus of variations*, Annals of Mathematics, vol. 28 (1927), pp. 153–170.

§ The set of points x at which one or more of the functions y_i fails to have a finite derivative may be neglected.

consisting of all admissible curves in K is either closed or empty.

We replace hypotheses (I), (II) and (III), pages 158 and 167 of the paper cited, by the following:

(I*) $E(x, y, y_0', y') \geqq 0$ for all points (x, y_0, y_0') and (x, y, y') in R^*.

(II*) For every bounded set A' contained in A there exist constants $M>1$ and $\alpha>1$ such that $f(x, y, y')>\|y'\|^\alpha$ for all sets (x, y, y') in R^* having (x, y) in A' and $\|y'\|>M$.

(III*) For every bounded set A' in A and every number $L>0$ there exists a number $\rho>0$ such that, if C is an admissible ordinary curve having at least one point in A' and at least one point outside the sphere S_ρ of radius ρ, then the integral

$$I_C = \int_C f(x, y, y')dx > L.$$

With these weakened hypotheses, the conclusions of Theorems 1, 2, 3 and 4 are still valid for classes of admissible curves.

Criteria which are sufficient to ensure the fulfillment of hypothesis (III*) may be selected from the following. In order to state these additional conditions it is convenient to change the notation. Let $t+s=k$, and

$$I = \int f(x, y_1, \cdots, y_t, z_1, \cdots, z_s, y_1', \cdots, y_t', z_1', \cdots, z_s')dx,$$

$$g_\mu(x, y, z) = 0, \qquad (\mu = 1, \cdots, m < t), \tag{1'}$$

$$z_\sigma' = h_\sigma(x, y, z), \qquad (\sigma = 1, \cdots, s). \tag{2'}$$

Let $\|yz\|$ = the greater of $\|y\|$, $\|z\|$.

(IV) The maximum difference of abscissas of points in the domain A is finite and equal to D.

(V*) The integrand $f(x, y, z, y', z')$ has a finite lower bound in the domain R^*.

(VI*) There exist constants $M>0$ and $\alpha \geqq 1$ and a function $\phi(y, z)$ continuous for all (y, z) such that

$$\phi(y, z) > 0 \tag{i}$$

for $\|y\| \geqq M$;

(ii)
$$\lim_{\|yz\|\to\infty} \|yz\|^{\alpha}\phi(y, z) = +\infty;$$

(iii)
$$f(x, y, z, y', z') \geqq \|y'\|^{\alpha}\phi(y, z)$$

for all (x, y, z, y', z') in R^* with $\|y\| \geqq M$, $\|y'\| \geqq M$.

(VII*) There exist positive constants c, d, α, β, such that

$$f(x, y, z, y', z') = \Phi(x, y') - \Psi(x, y, z, z'),$$

where

$$\Phi(x, y') \geqq c\|y'\|^{1+\alpha+\beta} - d, \quad \Psi(x, y, z, z') \leqq d\|yz\|^{1+\alpha} + d,$$

for all (x, y, z, y', z') in R^*.

(IX*) There exists a constant Γ such that on every admissible curve, and for every x and x_1,

$$\|z(x)\| \leqq \Gamma[\max\|y(\xi)\| + \|z(x_1)\| + 1].$$

(VI**) Same as (VI*), except that the function ϕ is independent of the z_σ, and properties (i) and (ii) do not involve the z_σ.

(IX**) For every number q there is a number Λ_q such that $|h_\sigma(x, y, z)| \leqq \Lambda_q$ for every (x, y, z, y', z') in R^* with $\|y\| \leqq q$.

Condition (III*) is implied by (a) (IV), (V*), (VI*), (IX*); or by (b) (IV), (V*), (VI**), (IX**); or by (c) (IV), (VII*), (IX*). For the proofs of these statements, I shall indicate the alterations necessary in the proofs given on pages 168–170 of the paper cited.

Let K^* be a class of admissible ordinary curves each of which has at least one point in the bounded set A'. Let $p>1$ be greater than the upper bound of $\|yz\|$ in the set A'. Then if $\|y\| \leqq q > p$ on a curve of K^*, also $\|z\| \leqq 3\Gamma q$ if (IX*) holds, and $\|z\| \leqq \Lambda_q D + p$ if (IV) and (IX**) hold. In either case, if $\eta_C \equiv$ maximum $\|y\|$, $\zeta_C \equiv$ maximum $\|yz\|$, on a curve C of the class K^*, then η_C becomes infinite with ζ_C. Let C be an arc of the class K^* having maximum $\|y\| \equiv Y > 2p$, and let $\bar{C}$ be a sub-arc having $\|y\| = Y$ at one end point, $\|y\| = p$ at the other end point, and $p < \|y\| < Y$ at interior points. If the conditions of group (a) hold, let ϕ_q be the minimum of $\phi(y, z)$ for $p \leqq \|y\| \leqq q$, $\|z\| \leqq 3\Gamma q$. If the conditions of group (b) hold, let ϕ_q be the minimum of $\phi(y)$ for $p \leqq \|y\| \leqq q$.

A few examples follow.

(1) $f = (zy' - yz')^2 - x^2, \qquad g = x^2 + y^2 + z^2 - 1 = 0,$

the region A is that included between $x=e-1$ and $x=1-e$, $0<e<1$.

$$f = [d^k z/dx^k]^{2n}, \tag{2}$$

where n and k are positive integers.

$$f = \frac{y'^4}{1+z^2}, \quad z' = y, \quad \text{or} \quad z' = (1+y^2)^{1/2}. \tag{3}$$

$$f = \frac{y'^2}{(1+z^2)^{1/2}}, \quad z' = y, \quad \text{or} \quad z' = (1+y^2)^{1/2}. \tag{4}$$

$$f = \frac{y'^4(1+z^2)}{1+y^2}, \quad z' = e^y. \tag{5}$$

$$f = y'^2 - y - z, \quad z' = (1+y^2)^{1/2}. \tag{6}$$

$$f = y'^2 - z'^2, \quad z' = h(x, y, z), \tag{7}$$

where h is a bounded function.

$$f = y'^4 - z'^3, \quad z' = h(x, y, z), \tag{8}$$

where

$$|h(x, y, z)| < \Gamma[|y| + 1].$$

In each example, a minimum exists in the class of all admissible ordinary curves joining two fixed points, provided that class is not empty. In Example (2), we may suppose the values of z and its first $k-1$ derivatives given at the end points. In examples (7) and (8), the function $h(x, y, z)$ should be substituted for z' in the integrand f before computing the Weierstrassian E-function.

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