ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THIS SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

1. Dr. D. C. Lewis: *Infinite systems of differential equations with applications to partial differential equations.*

Lichtenstein showed (Journal für Mathematik, 1927) how existence theorems for certain non-linear partial differential equations of hyperbolic type with initial and boundary conditions could be made to depend upon existence theorems for certain infinite systems of ordinary differential equations. In this paper analogous results are obtained for partial differential equations of less restricted type and for more general initial conditions. The gain in generality is principally due to a natural extension of the notion of a solution of a partial differential equation. Such extensions have also been considered for linear equations by N. Wiener, and for Laplace's equation by M. Bôcher. The author proposes the further study of a type of infinite systems, apparently not hitherto considered, of which the following system, arising from the consideration of a stretched elastic string constrained to vibrate transversally, is an example: $d^2x_\alpha/dt^2+\partial V/\partial x_\alpha=0$, $\alpha=1, 2, \cdots$, where $V=(1/2)\sum_{\alpha=1}^{\infty}\alpha^2x_\alpha^2+(\pi/48)\cdot(\sum_{\alpha=1}^{\infty}\alpha^2x_\alpha^2)^2+(\pi/96)\sum_{k=1}^{\infty}(\sum_{\alpha=1}^{k-1}\alpha(k-\alpha)x_\alpha x_{k-\alpha}+2\sum_{\alpha=1}^{\infty}\alpha(k+\alpha)x_\alpha x_{k+\alpha})^2$. (Received November 23, 1932.)

2. Dr. C. E. Rhodes: *Concerning the double Poisson integral and its derivatives.*

The single Poisson integral defines a function, $F(r, x)$, throughout the interior of the unit circle, which, under proper conditions, approaches an arbitrary given function, $f(x)$, along the boundary. Fatou has established (Acta Mathematica, vol. 30, pp. 335–400) certain sufficient conditions that the partial derivatives F_x and F_{xx} should approach respectively $f'(x)$ and $f''(x)$, as the point (r, x) approaches the boundary of the circle. The present paper considers the analogous situation for the function $F(r, \rho, x, y)$ defined by the double Poisson integral. Sufficient conditions that the partial derivatives F_x, F_y, F_{xy}, F_{xx}, F_{yy} shall approach respectively f_x, f_y, f_{xy}, f_{xx}, f_{yy}, as r and ρ approach 1, are found. In order to insure the proper behavior of the double integral over a cross-region about the point under consideration, an additional condition is used which has no analogy in the case of the single Poisson integral. (Received November 28, 1932.)

3. Dr. Mildred M. Sullivan: *Newtonian potentials as functionals of the masses.*

A Newtonian potential and its derivatives may be regarded as functionals of position, the distribution bearing surface or volume, and the density of the distribution. In a paper to appear in the January number of the Transactions of this Society, Dr. Sullivan has discussed the continuity of these functionals with respect to position alone, the masses being fixed, and has employed a simple condition due to Dini which is more general than the usual Hölder condition. The present paper, which was written in collaboration with the late Professor Kellogg, consists in an application of this same condition to the question of continuity of the potentials and their derivatives with respect to variations of the densities and the surfaces or volumes involved. (Received November 28, 1932.)

4. Dr. W. J. Trjitzinsky: *A theory of quasi-analytic classes.*

The author considers indefinitely differentiable classes C suitably defined by means of certain general types of infinite series. Application of the Schmidt theory of systems of linear equations in infinitely many unknowns (Palermo Rendiconti, vol. 25 (1908)) allows complete and explicit treatment of certain fundamental questions concerning quasi-analytic classes. Moreover, extension is made to the hitherto not extensively developed theory of quasi-analytic functions of the complex variable. The author was originally set along the road to the above developments by some results, due to Carleman, concerning quasi-analytic functions defined by cosine series. Use is also made of several of the author's earlier papers on quasi-analytic functions (Annals of Mathematics, (2), vol. 30, pp. 526–546; vol. 32, pp. 623–658, pp. 659–685). (Received November 28, 1932.)

5. Mr. L. B. Robinson: *On equations in mixed differences.* Part III.

In two abstracts in this Bulletin (vol. 32, pp. 316 and 590) the author has discussed the equation (A): $u'(x)=\sum_{j=1}^{n}P_j(x)u(a_jx+\delta_j)+Q(x)$, an equation similar to but more general than the differential q-difference equation. When the above mentioned abstracts were published the convergence of the solutions of (A) in the neighborhood of certain values of the δ had not been established by the author. Now let us write (A) in a new form as follows: (B) $u_x'(\delta, x)=\sum_{j=1}^{n}P_j(x)u(\delta, a_jx+b_j\delta)+Q(x)$. We can show that the solution of (B) depends on one arbitrary parameter. Also if $Q(x)$ and the $P(x)$ converge within the circle C then the solution converges so long as x and $a_jx+b_j\delta$ remain in C. (Received November 21, 1932.)

6. Professor Harry Bateman: *Some applications of Murphy's theorem.*

In 1830 Murphy gave a formula expressing the integral of a function between the roots of two equations $F(x)=0$ and $G(x)=0$ as an integral involving the logarithm of the ratio of $F(x)$ to $G(x)$. The present author shows that when this integral is regarded as a contour integral it is useful for the discovery of

integrals with variable limits such that both the integrand and the integral itself satisfy a given linear partial differential equation. The case in which the integrand is not uniform within the contour is also considered. (Received November 22, 1932.)

7. Professor F. R. Bamforth and Mr. H. H. Alden: *Concerning the uniqueness of solution curves of* $dy/dx=f(x, y)$.

The uniqueness of solution curves of $dy/dx=f(x, y)$ through a given interior point of the region of definition of $f(x, y)$ which is assumed to be a real, single-valued function of its arguments has been discussed by many writers in recent years. So far, with the exception of Kunugi and Yosie, all these writers have studied only sufficient conditions for uniqueness. This note is an outgrowth of a search for necessary conditions which do not assume $f(x, y)$ continuous as do those of the above mentioned authors. An example is given which shows that no one of the sufficient conditions which the authors of this note have found in the literature is necessary, and indicates that necessary conditions of simple form, if such exist for a general function $f(x, y)$, lie much deeper than do the sufficient conditions found until now. Furthermore, it is shown that, for the system $dy/dx=f(x, y, z)$, $dz/dx=g(x, y, z)$, the function $f(x, y, z)$ may be discontinuous at every point of its region of definition and yet there may be a unique solution through every point. (Received November 25, 1932.)

8. Dr. Frances T. Cope: *Formal solutions of irregular linear differential equations.*

The existence of a complete set of formal solutions of the general homogeneous linear differential equation with rational coefficients,

$$(1) \qquad a_0(x)y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \cdots + a_n(x)y(x) = 0, \ (a_0(x) \not\equiv 0),$$

was first proved by Fabry (Thèse, Paris, 1885). The corresponding existence theorem for an equation of the form (1) in which the coefficients are merely required to be formal series in descending powers of $x^{1/p}$, p being a positive integer, is here proved by a method developed by Birkhoff (Acta Mathematica, vol. 54 (1930), pp. 205–246) for the study of the analogous linear *difference* equation. This method consists of first reducing the original question to that of the formal reducibility of such an equation and then showing that any such equation is reducible if the "basic integer" p is properly chosen. A résumé is also given of the fundamental definitions and theorems of the algebra and differential calculus of the formal solutions of such differential and difference equations. Applications of the existence theorem and a proof of the converse are developed. These applications include general theorems on formal reducibility and formal equivalence at infinity, analogous to those given by Birkhoff for the linear difference equation. (Received November 19, 1932.)

9. Dr. C. H. Dix: *The length of closed level curves of a harmonic function.*

Let the level curve C^r correspond to the circle C_r under a conformal transformation of the region between two level curves $u(x, y)=c_1$ and $u(x, y)=c_2$

on an annular ring. Let $l(C^r)$ denote the length of C^r. Then we have the following result: $l(C^r)/l(C_r)$ is a convex function of $\log r$. This follows by an application of a theorem of F. Riesz (Acta Mathematica, vol. 48 (1926), p. 338). (Received November 25, 1932.)

10. Professor L. M. Graves: *On the existence of the absolute minimum in problems of Lagrange.*

This note points out that the proofs given in the author's paper *On the existence of the absolute minimum in space problems of the calculus of variations* (Annals of Mathematics, (2), vol. 28 (1927), pp. 153–170) are applicable under suitably *weakened hypotheses* to problems of Lagrange in which the side conditions consist of equations in finite form and differential equations of the special form $y'_\alpha = h_\alpha(x, y_1, \cdots, y_k)$. Calculus of variations problems in which the integrand involves higher derivatives are reducible to Lagrange problems of this type, as are also isoperimetric problems in which the integrands of the integrals to be kept constant depend only on the coordinates. (Received November 15, 1932.)

11. Professor E. L. Mackie: *On minimizing the quotient of two definite integrals.*

In the Annals of Mathematics, (2), vol. 30 (1926) p. 393, the problem was considered of finding an arc which minimizes the quotient of two definite integrals while keeping a third integral constant. This was carried through as a problem of Mayer. In the present paper, for the same initial problem the standard method of the calculus of variations is employed in finding the first necessary condition for the required arc, and the function to which the Euler equation is applied is shown to be a linear combination of the integrand functions of the three definite integrals, the coefficients of these functions being constants. (Received November 23, 1932.)

12. Professor Florence M. Mears: *Summability for absolutely convergent series.*

This paper deals with methods of linear transformation by square matrices. It determines the set A of matrices such that every absolutely convergent series, or associated sequence, is transformed into a convergent sequence; the subset of A for which the sum of the transformed series is equal to the sum of the original series; the set B of matrices such that every absolutely convergent series, or associated sequence, is transformed into a sequence whose corresponding series is absolutely convergent. (Received November 23, 1932.)

13. Professor I. M. Sheffer: *An extension of Leibnitz's formula for product-differentiation to differentiation and integration for all real orders.*

In the Transactions of this Society, vol. 32 (1930), E. L. Post has given an extended theory of "generalized differentiation," working with a general differential operator $f(D)$. Among other results, he has generalized the Leibnitz formula for a product $f(D)u(x)v(x)$. An interesting subcase is that where

$f(D)=D^a$, a any real number, giving the Leibnitz formula for fractional integration and differentiation. This subcase appears to be of sufficient interest in itself to warrant a simplified treatment, independent of Post's general theory; the present note serves this purpose. At least two writers previous to Post have considered this subject, but formally rather than rigorously. (Received November 21, 1932.)

14. Dr. A. H. Smith: *On the summability of derived series of the Fourier-Lebesgue type.*

Bosanquet and Linfoot (Journal of the London Mathematical Society, vol. 6 (1931), pp. 117–126; Quarterly Journal of Mathematics, Oxford series, vol. 2 (1931), pp. 207–229) introduced a regular method of summation which is weaker than that of the Cesàro means of any order $\alpha>0$, and applied it to the Fourier series and its conjugate series of a function of class L. The present paper extends the former method of summation to the case of the rth derived Fourier series of a function of this type and to the case of the first derived Fourier series and its conjugate series when the function is restricted to be of bounded variation. (Received November 29, 1932.)

15. Dr. W. J. Trjitzinsky: *The general case of integro-q-difference equations.*

The author proves the existence (in the complex plane) of analytic solutions of equations of the form $L(y)=V(y)$. Here L is an operator, with analytic coefficients, of the type occurring in linear homogeneous q-difference equations of order n. On the other hand, V is an integral operator of the Volterra type. While the treatment of this problem (for the complex plane) has been given by C. R. Adams when there is only one characteristic equation, the present author treats the general case. This paper is a sequel to a paper by the author on linear q-difference equations, which will appear in the Acta Mathematica. (Received November 28, 1932.)

16. Miss Mabel Griffin: *Invariants of Pfaffian systems.*

Let Ω be the exterior product of r independent Pfaffians, and let M_i be the set comprising all products of Ω by i of the derived forms. Each set M_i is invariant for the corresponding Pfaffian system of equations S. The vanishing of M_1 is known to be the condition for passivity of S. The present paper interprets the vanishing of M_2 by means of the notion of a primitive system of S, that is, a system whose derived system is S. The sets M_i are shown to give rise to certain arithmetical invariants of S. They are also used to express a necessary and sufficient condition that any S be completely separable, that is, be equivalent to r equations expressible in terms of r non-overlapping sets of independent variables. (Received November 29, 1932.)

17. Mr. Y. Y. Tseng: *Expansions according to a given system of functions.*

In this paper a non-Hilbertian space is proposed for study. It is characterized abstractly by a simple set of properties, but it can have more than a

denumerable infinity of dimensions. We impose no restriction on the given system. (1) The functions need not be mutually independent, and (2) the number of functions may be non-denumerable. In a space not known to be complete the Bessel inequality and formula for a best approximation are obtained. We establish the equivalence of Fourier and Parseval equalities. If the space is complete, then, by means of the Fourier coefficients, every function can be expanded according to the given system, provided the function lies in a certain subspace connected with that system. Two well known theorems still hold: the Parseval equality and the Riesz-Fischer theorem. It may be noted that the abstract space considered here includes Hilbert space, E. H. Moore's modular space, and others, as special cases. We make use of some machinery developed in Moore's second theory of general analysis (not yet published), and the method is entirely direct, involving no orthogonalization process. (Received November 29, 1932.)

18. Mr. Y. Y. Tseng: *The characteristic value problem of Hermitian functional operators in a non-Hilbertian space.*

The problem studied in this paper is the one formulated by von Neumann and solved by von Neumann and by Stone for (separable) Hilbert space. However, we consider here, as in the preceding paper (abstract No. 39–1–17) a quaternionic, non-separable, complete abstract space. Equivalent conditions of completeness and compactness of a space are discussed. We have also the useful theorems on orthogonal decomposition and representation of linear functionals. Explicit formulas for projections relative to sum- and intersection-spaces are found. We show that the problem always admits a unique solution, if and only if the operator is self-adjoint or hypermaximal. We establish a number of properties of the spectra, resolvents, separations, and resolutions of the identity. The discontinuities of E_t can be non-denumerable. Orthogonal decompositions of E_t are made with reference to its continuity character. For every self-adjoint operator there exist three orthonormal characteristic systems (including differentials). In terms of these, the initial operator and the identity operator can be expanded into general series by means of strict functional Hellinger integrals. The results given here contain, indeed as special instances, those for self-adjoint operators in Hilbert space and those for modular Hermitian matrices in general analysis. (Received November 29, 1932.)

19. Dr. S. S. Wilks: *A generalized sampling distribution for the multiple correlation coefficient.*

The sampling distribution of the multiple correlation coefficient has been found by R. A. Fisher (Proceedings of the Royal Society, A, vol. 121 (1928), pp. 654–673) in samples of N individuals from an n-variate normal population for the two extreme cases, (1) that in which all of the variates are subject to sampling variation, and (2) that in which the $n-1$ independent variates have values which are free from sampling variation, while the dependent variate is normally distributed. The distributions for (1) and (2) are denoted by (A) and (C) respectively, and were obtained by entirely different methods. In this paper the more general problem is considered in which each of an arbitrary number t

of the independent variates has a set of N values which are constant for all possible samples while the remaining $n-t-1$ independent variates and the dependent variate are normally distributed. The distribution of the multiple correlation coefficient for this case is found and is shown to reduce to Fisher's (A) and (C) distributions respectively when $t=0$ and $n-1$. (Received November 18, 1932.)

20. Professor F. D. Murnaghan: *On the parametric representation of orthogonal and unitary square matrices.*

The matrix formulas $\boldsymbol{U}=\exp(i\boldsymbol{H})$, $\boldsymbol{O}=\exp(-\boldsymbol{S})$, where $\boldsymbol{U}, \boldsymbol{H}, \boldsymbol{O}, \boldsymbol{S}$ are, respectively, unitary, Hermitian, orthogonal, skew-symmetric matrices, are used to give parametric representations of orthogonal and unitary matrices. The case of four-rowed matrices is treated in some detail. (Received November 25, 1932.)

21. Professor Leonard Carlitz: *On a theorem of higher reciprocity for polynomials in a Galois field.*

This paper contains a new proof of the theorem of reciprocity of index p^n-1 expressed by $(Q/P)=(-1)^{\rho\nu}(P/Q)$, where P and Q are primary irreducible polynomials in $GF(p^n)$, of degree ν and ρ respectively (American Journal of Mathematics, vol. 54 (1932), pp. 39–50). The proof is modeled after Zeller's proof of the ordinary quadratic reciprocity theorem. (Received November 28, 1932.)

22. Mr. J. O. Blumberg: *Properties of a certain type of continued fractions.*

Gauss has shown that the quotient of two "consecutive" hypergeometric functions can be easily represented by a continued fraction of the second class. In the present paper we study at some length the special case of Gauss's continued fraction $J_1(2x)/J_0(2x)$. By means of recursion formulas a triangular numerical table is constructed from which we obtain numerous number theory relations connected with this continued fraction. For some of these properties geometric interpretations are presented. Finally we point out corresponding properties of more general cases of Gauss's continued fraction. (Received November 25, 1932.)

23. Professor J. B. Coleman: *The Jacobian algorithm for periodic continued fractions as defining a cubic irrationality.*

In the Jacobian continued fraction algorithm, when the partial quotients become periodic, certain ratios of the convergents approach limits. These limits have been shown to be rational linear functions of the principal root of a characteristic equation of the form $\rho^3-M\rho^2+N\rho-1=0$. In this paper the characteristic equation is proved to be irreducible by means of two types of recursion formulas, in terms of which M and N are expressed. (Received November 22, 1932.

24. Professor B. W. Jones: *The transformations effecting the reduction of positive quaternary quadratic forms.*

To date there has been found no reduction of positive quaternary quadratic forms which defines a *unique* form by inequalities on the coefficients. L. Charve has given inequalities which make the sum of the literal coefficients a minimum for the class. We call such a form "semi-reduced." There are usually many semi-reduced forms in a class of equivalent forms. This paper shows that all transformations taking a semi-reduced form into a semi-reduced form are of one or more of four or five types multiplied on the left by certain automorphs. This brings the problem of reduction within reach. (Received November 23, 1932.)

25. Professor Leonard Carlitz: *Some formulas in additive arithmetic.*

In this paper asymptotic expressions are found for a class of sums of the form $\sum\alpha(n-x^2)$, the sum extending over all positive x, $x^2<n$. The method used is quite elementary. By properly specializing the arithmetic function, and using certain known results, we determine the leading term in the asymptotic expression for the number of positive integral solutions of $n=x^2+m_1+\cdots+m_\nu$, where p^{s_i} does not divide m_i, and the s_i are arbitrary integers $\geqq 2$. (Received November 28, 1932.)

26. Dr. R. D. James (National Research Fellow): *On Waring's problem for odd powers.*

Let $G(k)$ be the least value of r such that all sufficiently large numbers are sums of r non-negative kth powers. In this paper a new method is used to obtain an upper bound for $G(k)$ when k is odd and $\geqq 5$. These values are considerably better than those previously known. (Received November 26, 1932.)

27. Miss Clarice Van Hook: *Pairs of generators of groups of degree not greater than eight.*

Aside from the cyclic groups of odd order, there are sixteen groups which can be written transitively on eight or fewer letters that cannot be generated by two operators of which one is of order two. Pairs of such generators are given in those cases where they exist for the groups of degree that cannot be written transitively of lower degree. (Received November 25, 1932.)

28. Professor C. G. Latimer: *On the class numbers of a cyclic field and a sub-field.*

Let F be an algebraic field which is cyclic and of degree e with respect to the rational field. Assume that the discriminant of every sub-field, not rational, of F contains a prime factor not a divisor of e. If F_1 is a sub-field of F and if h, h_1 is the number of classes of narrowly equivalent ideals in F, F_1, respectively, it is shown that h is divisible by h_1. (Received November 28, 1932.)

29. Professors Jesse Douglas and Philip Franklin: *A step-polygon of a denumerable infinity of sides which bounds no finite area.*

In this paper the authors construct an example, simpler than that previously given by them (Transactions of this Society, vol. 33 (1931), p. 321), of a Jordan curve in space such that every surface bounded by this curve has infinite area. The curve is made up of a denumerable infinity of straight line segments, each parallel to one of the coordinate axes. (Received November 26, 1932.)

30. Professor A. J. Kempner: *On the summation of infinite series.*

Given a Laurent series $\sum a_n(z-\alpha)^n=f(z)$, convergent for $r_1<|z|<r_2$, and a rational function $R(n)=P(n)/Q(n)$ with complex (including real) coefficients, then the sum of the series $\sum R(n)a_n(z-\alpha)^n=u(z)$, convergent in $r_1<|z|<r_2$, is expressible by means of the operator $\overline{D}=z(d/dz)$, from the equation $\sum R(n)a_n(z-\alpha)^n=R(\overline{D})f(z)$. The function $u(z)$ satisfies the differential equation $(P(\overline{D})/Q(\overline{D}))f(z)=u(z)$, $Q(\overline{D})u(z)=P(\overline{D})f(z)=g(z)$, say, and this differential equation can be solved by any one of several methods; or, $R(\overline{D})f(z)$ may be immediately reduced to differentiations and integrations. Throughout, proper precautions must be taken concerning values of n for which $R(n)=\infty$. Incidentally, it is seen that a differential equation $R(\overline{D})u(z)=f(z)$, provided $f(z)$ has an expansion $f(z)=\sum a_n(z-\alpha)^n$, has a solution $u(z)=\sum[R(n)]^{-1}\cdot a_n(z-\alpha)^n$. Similar results hold for Dirichlet series $f(s)=\sum c_n e^{k_n s}$ and $\sum R(k_n)c_n e^{k_n s}$, with the operator $D^2=d/ds$, and more special results for trigonometric series. Extension to series in more than one variable is feasible. (Received December 29, 1932.)

31. Professor H. B. Phillips: *The potential transformation.*

The ordinary potential of a space distribution may be regarded as a transformation which changes a bounded and integrable space density into a function with continuous first derivatives. In this paper the author shows that when valid for all space such a transformation leaves many important relations of scalar and vector functions unchanged. By the use of integrals instead of the usual derivatives the inverse transformation is defined in a form free from certain restrictions ordinarily imposed to insure the existence of second derivatives. (Received November 26, 1932.)

32. Professor I. M. Sheffer: *An aspect of the theory of linear differential equations.*

It occurred to the author that a method he was using to examine some functional equations might be applicable to a phase of linear differential equation theory. It is proposed to give a treatment of this application. First an existence theorem (in the complex domain) is obtained, not by a method of approximations or use of power series but rather by a direct search for the inverse of the differential operator. The solution obtained is expressed as an infinite series which is of interest, not only because of the character of the re-

gion of convergence (which is the interior of a "polygon"), but also because it displays explicitly (although not in "closed" form) the dependence of the solution on the coefficients of the equation. It is believed that the results concerning the series which defines the solution are new; from these are obtained many conclusions which will be recognized as classic. (Received November 21, 1932.)

33. Dr. I. J. Schoenberg and Dr. Wladimir Seidel: *On linear operations in linear metric spaces.*

F. Riesz (Mathematische Annalen, vol. 69 (1910); Annales de l'École Normale Supérieure, (3), vol. 28 (1911); *Les Systèmes d'Équations Linéaires à une Infinité d'Inconnues*, Paris, 1913) and E. Helly (Wiener Sitzungsberichte, IIa, vol. 121 (1912); Monatshefte für Mathematik und Physik, vol. 31 (1921)) have developed methods with the aim to determine the most general linear operations in some special metric spaces. In his recent book *Théorie des Opérations Linéaires*, Warsaw, 1932, S. Banach develops a general method based on prior results of his own and H. Hahn's (Journal für Mathematik, vol. 157). In the present paper, a method is developed which is an independent synthesis of the older methods of Riesz and Helly. This method is more special in its general aspects than the methods of Banach, and for precisely this reason seems to be well adapted to the problem of the determination of the most general linear operation in special linear metric spaces. It permits us to re-establish in a simple way all the known results in this direction and to derive a score of new ones. (Received November 28, 1932.)

34. Professor J. A. Shohat: *On the continued fractions associated with, and corresponding to,* $\int_a^b [p(y)/(x-y)]dy$.

The author gives a discussion of the asymptotic behavior of the distant elements in the continued fractions associated with, and corresponding to, $\int_a^b [p(y)/(x-y)]dy$ ((a, b) finite), for the case of $p(x)$ vanishing over a set of points $\subset (a, b)$ of *positive measure*. The discussion is based on the relation of certain extremal polynomials to the Tchebycheff polynomial ($=T$-polynomial) the least deviating from zero on a given point set. The results of Faber concerning T-polynomials and the notion of "transfinite diameter" (Fekete) for point sets play an important role in the present discussion. (Received November 18, 1932.)

35. Dr. Hassler Whitney (National Research Fellow): *Analytic extensions of differentiable functions defined on closed sets.*

Let E be euclidean n-space, and A, a closed subset of E. Let $f=f_{0\dots0}$ be defined and m-fold differentiable ($m\geqq 0$) in A in the following sense: There are functions $f_{k_1\dots k_n}(k_1+\cdots+k_n\leqq m)$ defined in A and such that if x and x' are in A, then $f_{k_1\dots k_n}(x')=\sum f_{k_1+l_1\dots k_n+l_n}(x)(x_1'-x_1)\cdots(x_n'-x_n)/(l_1!\cdots l_n!)$ $+R_{k_1\dots k_n}(x'; x)$ (in $\sum$, $l_1+\cdots+l_n\leqq m-k_1-\cdots-k_n$), where for every $\epsilon>0$ there is a $\delta>0$ (independent of the point x) such that $|R_{k_1\dots k_n}(x'; x)|$ $<r^{m-k_1-\cdots-k_n}\epsilon$ if the distance r from x' to x is $<\delta$. Then there is a function F defined in E and having continuous partial derivatives through the mth order there, analytic in $E-A$, and coinciding with f in A. A similar theorem holds if $m=\infty$. (Received December 15, 1932.)

36. Professor W. S. Kimball: *The ellipsoidal viscosity distribution function.*

This paper continues the work of the author and his students in removing probability as the foundation stone of statistical mechanics and substituting Newtonian mechanics and elasticity theory. The geometrical weight method is used, and the distribution is an exact solution of Boltzmann's equation. It involves a parameter that varies with molecular position along free paths, and includes a distribution immediately *before* impact that is the mechanical image of that just *after* impact. Elasticity theory relative to homogeneous strain, simple shear, and the strain ellipsoid is taken over into velocity space. The distribution functions before and after impact are obtained directly from the Maxwell distribution by subjecting its equal probability spheres to a simple shear that yields equal probability ellipsoids, being also the strain ellipsoids of the distribution. Distortion but no expansion in velocity space is involved, whereby the entropy and strain are the same as for equilibrium conditions at that temperature. The elastic modulus is again the simple pressure formula, the isothermal bulk modulus for perfect gas. (Received November 23, 1932.)

37. Dr. G. B. Price: *On the Strömgren-Wintner natural termination principle.*

Poincaré developed the subject of analytic continuation of periodic orbits. Recently it has been considered in two papers by Wintner (Mathematische Zeitschrift, vol. 34 (1931), pp. 321–402). He has proved what we shall call the Strömgren-Wintner natural termination principle for groups of periodic orbits. The only example of the principle offered so far is the restricted problem of three bodies, a problem so complicated that the groups can be studied only by numerical integration of the equations of motion. The present paper furnishes a simple example which can be treated mathematically. Wintner's remarks in the proof of the natural termination principle give the impression that his results are in disagreement with those of Poincaré. The present example sets forth the results of both investigators, and shows that there is no essential disagreement. The difference is in the point of view. Finally, this paper adds further information about a class of dynamical systems on surfaces of revolution previously studied by the author (American Journal of Mathematics, vol. 54 (1932), pp. 753–768). (Received November 21, 1932.)

38. Professor C. H. Forsyth: *Interpolation with rationally centered frequencies.*

One of the greatest obstacles to satisfactory interpolation with equi-spaced frequencies, or what are interpreted as frequencies, has heretofore been the practical necessity of associating each frequency with the middle of the range of the corresponding class. The present author has followed the suggestion that the interpolation formula chosen be employed not only to interpolate but also for a preliminary rational approximation of the most appropriate association, and found corresponding simple modifications for a few of the most used interpolation formulas, which will accomplish the purpose described above. These simple modifications prove to have other interesting and valuable virtues also. (Received November 28, 1932.)

39. Mr. J. M. Clarkson: *Some involutorial line transformations.*

Consider a cubic regulus and a plane pencil of lines. An arbitrary line meets three generators of the regulus and one line of the pencil. These four lines have in general one other transversal. The transformation among the Plücker coordinates of the two transversals is of order three. The invariant complex and the singular elements of the transformation are discussed, and the results considered as of a point transformation in five-space. The transformation determined by the transversals of four generators of a quartic regulus is found. Another involutorial line transformation is determined by joining an arbitrary point in one plane to an arbitrary point in a second plane, there being a harmonic homology given in each plane, and the image of the given line being the join of the points which are the images by the two homologies of the given points. (Received November 21, 1932.)

40. Professor T. R. Hollcroft: *The general web of surfaces and the involution defined by it.*

The general web of surfaces is a triply infinite linear system of surfaces of order n. These surfaces are in (1,1) correspondence with the planes of space. This correspondence defines a general space involution whose characteristics are determined. The web contains two doubly infinite, four infinite, and six finite systems of surfaces satisfying respectively one, two, and three invariant conditions associated with singularities or contacts of the surfaces. The jacobian is the locus of the conic nodes and contacts of the two doubly infinite systems. On the jacobian are four curves, the loci respectively of two conic nodes, one binode, two contacts, one stationary contact. The jacobian also contains six finite sets of points which are respectively sets of three conic nodes, one conic node and one binode, one binode whose axis has four-point contact, three contacts, two contacts of which one is stationary, one tacnodal contact. The characteristics of each of the three curves and the number of surfaces in each of the six finite systems are obtained. Webs of quadrics have been studied previously. Certain characteristics of quadric webs do not result on substituting 2 for n in the general formulas, since quadric webs contain degenerate surfaces, a situation that does not arise for $n>2$. (Received November 28, 1932.)

41. Mr. J. C. Knipp: *Certain relations between the Frégier curve and the evolute.*

All chords of a conic which intercept a right angle at a fixed point P of a conic pass through a common point Q; the latter point is known as the Frégier point, and its locus as P moves around the conic is the Frégier curve. This paper considers the relation of this curve to the evolute. The two curves are found to be tangent in four points, and these points are shown to be exceptional for the Frégier curve in several ways. Special properties of the normals from these points to the corresponding conic are developed. (Received November 22, 1932.)

42. Professor Evelyn Carroll Rusk: *Cremona involutions defined by a pencil of cubic surfaces.*

In the American Journal of Mathematics, vol. 54, pp. 707–717, there is a discussion of the Cremona involution defined by means of a pencil of algebraic surfaces $|F_n|$ of order n, having a straight line d to multiplicity $n-2$ and a simple, non-composite curve $\gamma_{4(n-1)}$ of order $4(n-1)$ as base curves. The present paper is restricted to cubic surfaces with a base line d and a residual γ_8 made up of all possible components. The interest lies in the distribution of the parasitic lines, the reduction of the order of the transformation, and the contact of the various principal surfaces along the line d. The generalization applied to a pencil of algebraic surfaces $|F_n|$ offers no new difficulties. (Received November 28, 1932.)

43. Dr. A. E. Staniland: *On the Segre curved four-space representation of the plane of two complex variables.*

Conspicuous among real representations of the complex plane is that of Segre, which employs as representatives of the complex points the real points of a curved space of four dimensions and order six, S_4^6, within an S_8. In the present paper, the author determines the analytic rotations of S_8, i.e., those rotations which transform surfaces representing analytic functions into surfaces representing analytic functions. These rotations are obtained in finding the rotations of S_8 which leave S_4^6 invariant as a whole. Surprisingly, the two classes of rotations are identical. The author classifies the rotations in two ways. First, they are subdivided into ∞^2 sets of ∞^2 members each, each of these sets being subdivided into ∞^1 sets of ∞^1 members each, the sets being characterized by definite effects upon quadric surfaces representative of the lines $Y=MX$. Second, the group is factored into products of types $T_1T_2T_4$, and $T_1T_2T_4$, where T_1, T_2, and T_4 are subgroups each of which leaves one or more of the coordinate axes invariant, and $\overline{T}_1=RT_1$, R being a pair of reflections, one in each of a certain pair of coordinate planes. (Received November 25, 1932.)

44. Dr. S. S. Cairns: *Polyhedral approximations to regular topological n-manifolds.*

A θ-set, in the present paper, will mean $(n+1)$ points which determine such an n-simplex that the angle between any bounding $(n-1)$-simplex and an edge incident with the opposite vertex exceeds $\theta>0$. Consider the triangulation (σ) of M_n announced in the abstract *On the triangulation of topological n-manifolds in $(n+r)$-space* (presented October 29, 1932) and the corresponding simplicial polyhedron, P_n (loc. cit.). For some positive values (ρ, θ), we can make (σ) arbitrarily fine and still require (a) that the vertices of each n-cell be a θ-set and (b) that $s_1>\rho s_1'$, for any 1-simplexes (s_1, s_1') of P_n. A homeomorphism is defined between M_n and P_n such that any i-cell σ_i of (σ) corresponds to the i-simplex, s_i, of P_n having the same vertices. Furthermore, given any positive ϵ, if (σ) is made sufficiently fine under conditions (a) and (b), then (1) the direction cosines of any tangent i-plane (*The direction cosines of a p-space in a euclidean n-space*, American Mathematical Monthly, November, 1932) to σ_i

differ by less than ϵ from the corresponding direction cosines of s_i, and (2) $PP' < \epsilon s_1$ where (P, P') are corresponding points and s_1 is the shortest 1-simplex of P_n. The n-dimensional volume of M_n is then the limit of that of P_n as the triangulation becomes increasingly fine. (Received November 28, 1932.)

45. Dr. P. M. Swingle (National Research Fellow): *Special types of higher dimensional closed sets.*

In this paper a closed set whose r-dimensional Betti number (R. L. Wilder, this Bulletin, vol. 38, p. 679) is zero is called an r-continuum, being for $r=0$ the ordinary continuum. Among these r-continua a continuous curve receives a generalization in what is called a $(j;r)$-continuum; a simple continuous arc in what is called a special Γ r-arc, where Γ is a special type C of complete cycle; and a simple closed curve in what is called a special simple closed r-continuum. A number of theorems are proved concerning these point sets which are generalizations of results well known for the case $r=0=j$. For instance, every special simple closed r-continuum is divided by a Γ of type C into two special Γ r-arcs having only the closure of Γ in common. Also the following generalization of a fundamental lemma of Knaster and Kuratowski (Fundamenta Mathematicae, vol. 2, p. 210, Theorem 6) is obtained: Let M and N be r-continua such that M contains N and $M-N=M_1+M_2$ separate. Then M_i+N $(i=1, 2)$ is an r-continuum. (Received November 28, 1932.)

46. Dr. L. M. Blumenthal and Mr. G. A. Garrett: *Determinantal theory of d-cyclic and pseudo d-cyclic sets of points.* Preliminary report.

The problem of the characterization of the euclidean n-dimensional space R_n among general semi-metric spaces has been shown by Menger (American Journal of Mathematics, vol. 53 (1931), pp. 737–745) to be intimately connected with the theory of the determinant denoted symbolically with first row 0, 1, and second row 1, $(p_ip_j)^2$, $(i, j=1, 2, \cdots, n)$. The investigations presented in the present report show that the characterization of the spherical n-dimensional space S_n is closely related to the theory of the determinant $|\cos \alpha_{ij}|$, $(i, j=1, 2, \cdots, n)$. For the case of the circle S_1, the several theorems characterizing d-cyclic and pseudo d-cyclic sets of points contained in two recent papers by L. M. Blumenthal (American Journal of Mathematics, vol. 54 (1932), pp. 387–396; 729–738) are expressed as new theorems concerning determinants of the above form. (Received November 21, 1932.)

47. Professor N. H. McCoy: *On complete independence of certain sets of postulates for fields.*

In this paper the author gives a method of combining simple systems so as to obtain a system with any desired character. The complete independence of certain sets of field postulates may then be established by exhibiting explicitly only a small number of systems. By means of only ten systems, a set of seven postulates for fields, essentially those given by Huntington (Transactions of this Society, vol. 4 (1903), p. 33) is shown to be completely independent. A set of five systems showing the ordinary independence of the five postu-

lates of Hurwitz (Annals of Mathematics, vol. 15 (1913–14), pp. 93–100) is also sufficient to establish the complete independence of these postulates. The complete independence of this latter set had been previously proved by Bernstein (Annals of Mathematics, vol. 23 (1921–22), pp. 313–316). (Received November 28, 1932.)

48. Dr. G. A. Hedlund: *Recurrent geodesics on any closed orientable surface of genus one.*

The existence of non-periodic recurrent geodesics on certain open surfaces of negative curvature has been proved by Morse. Birkhoff has indicated how the methods of Morse may be used to prove the existence of non-periodic recurrent geodesics on closed orientable surfaces of genus greater than one and of negative curvature. The property of negative curvature plays an essential part. In the case of a closed orientable surface of genus one, the curvature cannot be everywhere negative. Nevertheless it is possible in this case to prove the existence of a non-denumerable set of non-periodic recurrent geodesics. This result will appear in the Proceedings of the National Academy of Sciences. (Received November 23, 1932.)

49. Professor R. C. Bullock: *Non-conjugate osculating quadrics of a curve on a surface.*

It is the primary purpose of this paper to make a study of the projective differential geometry of a non-conjugate net of curves on a surface in ordinary projective space. This study is made by means of a pair of osculating quadrics, one determined by the tangents to three consecutive curves of one family of the non-conjugate net at a point of a general curve on the surface, and the other defined similarly for the other family of the net. Complete results as to the nature of the curve of intersection of the two quadrics thus associated with each point of the surface are obtained. A new necessary and sufficient condition that a net of curves on the surface be a conjugate net is found. Finally, a method of making a similar study of a net on a surface in metric space of three dimensions is indicated. (Received November 25, 1932.)

50. Professor Dunham Jackson: *The convergence of some non-linear processes of approximation.*

If $f(x)$ is a continuous function of period 2π which is everywhere positive, and if an approximating function is sought in the form $e^{T_n(x)}$, where $T_n(x)$ is a trigonometric sum of the nth order, with the specification that the integral of the square of the error (or of the mth power of the absolute value of the error) over a period shall be a minimum, the approximating function depends non-linearly on its parameters, and the detailed consideration of the problem involves certain complications not previously encountered in problems of similar character. With appropriate supplementary hypotheses, this paper demonstrates the existence of an approximating function satisfying the conditions laid down, for fixed n, and its convergence toward $f(x)$ as n becomes infinite. There is an analogous treatment of the corresponding problem of polynomial approximation. (Received November 17, 1932.)

51. Professor G. C. Evans: *Note on Poincaré's sweeping out process.*

The author removes some of the restrictions imposed in the treatment by de la Vallée-Poussin (Annales de l'Institut Henri Poincaré, 1931) of Poincaré's "méthode du balayage." In particular, no restriction is imposed on the original distribution of mass in the region, except that its total amount be finite, and less restrictive conditions of regularity are obtained. (Received November 15, 1932.)

52. Professor Marston Morse: *The tensor form of boundary problems in the calculus of variations.*

Weierstrass removed the limitations on the calculus of variations implied by the non-parametric form. The limitations implied by the use of a single coordinate system have likewise been removed by the use of tensor analysis, in so far as the Euler, Weierstrass, and Legendre conditions are concerned. It is not so with the general accessory boundary problem involving a parameter. When the problem is defined by means of overlapping coordinate systems, it is by no means a priori clear that such an accessory problem exists, still less that it has any invariant or geometric meaning. In this paper an accessory problem is set up in tensor form for the general boundary problem. It reduces to the ordinary accessory problem in non-parametric coordinates when the coordinates are suitably specialized. Characteristic roots and their indices appear as invariants, and hence may properly be considered as geometric in nature. Such problems as the Sturm-Liouville problems and their extensions in the quantum theory, in this relativistic setting appear as special problems in geometry. (Received November 21, 1932.)

53. Professor Henry Blumberg: *Exceptional sets.*

Beginning with an elementary theorem in point sets, the author derives, through a series of steps that easily suggest themselves, a comprehensive body of theorems relating to arbitrarily given sets and functions and asserting that certain associated sets are exceptional, for example, in the sense of cardinal number (denumerability), or of density (non-density, exhaustibility), or of measure (measure zero). There is thus effected a unification and an extension, along simple lines heretofore unobserved, of many known theorems, such as, for example, one of G. C. Young on Dini derivatives, or the author's general theorem on exceptional denumerable sets in Fundamenta Mathematicae, vol. 16 (1930), p. 18, or W. H. Young's results on the symmetry of arbitrary functions of one or more variables, or various results of the author on general functions involving exceptional sets that are exhaustible or of measure zero. (Received Novermber 25, 1932.)

54. Professor Tibor Radó: *An alternating process in the problem of Plateau.*

The problem of Plateau calls for a surface S: $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$, with the following properties: (1) $x(u, v)$, $y(u, v)$, $z(u, v)$ are harmonic for $u^2+v^2<1$; (2) they satisfy there the relations $E=G$, $F=0$; (3) these

three functions are continuous in $u^2+v^2\leqq 1$, and the equations $x=x(u,v)$, $y=y(u, v)$, $z=z(u, v)$ carry $u^2+v^2=1$ in a topological way into a given Jordan curve. The alternating process developed in the present paper consists in the construction of a sequence of surfaces which all satisfy (3), and satisfy (1) and (2) alternately. The sequence contains a subsequence converging uniformly toward a solution of the problem of Plateau. (Received November 29, 1932.)

55. Professor D. V. Widder: *The inversion of the Laplace integral and the related moment problem.*

The inversion of the Laplace integral $f(x)=\int_0^\infty e^{-xt}d\alpha(t)$ is analogous to the determination of the coefficients of a power series. In fact the integral reduces to a power series in e^{-x} if the function $\alpha(t)$ is a suitable step-function. There are two classical methods of determining these coefficients, one by means of a contour integral, the other in terms of the derivatives of the function represented. We thus expect two methods of inversion of the given integral. That involving contour integration was discovered by Riemann. That involving the derivatives of $f(x)$ was found by E. L. Post (Transactions of this Society, 1930) for the special case in which $\alpha(t)$ has a continuous derivative. It is the purpose of the present paper to treat the general case, and determine $\alpha(t)$ in terms of $f(x)$ and its derivatives. The explicit nature of the inversion formula makes it a powerful tool in applications of the Laplace integral. The moment problem $\mu_n=\int_0^1 t^n d\alpha(t)$ is treated in an analogous manner, and an explicit determination of $\alpha(t)$ in terms of the successive differences of the sequence μ_n is obtained. This inversion formula seems to be entirely new. (Received November 23, 1932.)

56. Dr. E. F. Beckenbach (National Research Fellow) and Professor Tibor Radó: *Subharmonic functions and minimal surfaces.*

The theory of analytic functions of a complex variable has been a guiding principle in the discussion of minimal surfaces since the time of Weierstrass; particularly, this statement applies to the recent developments in the problems of Plateau. Those theorems that are based on the fact that the product of a finite number of analytic functions is again an analytic function have heretofore escaped generalization. We offer now as a substitute for this the fact that if the coordinate functions $x(u, v)$, $y(u, v)$, $z(u, v)$ of a minimal surface are given in terms of isothermic parameters, then $\log (x^2+y^2+z^2)$ is a subharmonic function. This permits in particular the application of the so-called rotation method of analytic function theory to the theory of minimal surfaces. The discussion applies equally to the case of minimal surfaces in n-dimensional space. (Received November 30, 1932.)

57. Dr. Selby Robinson (National Research Fellow): *Pseudo-transitivity in finite and infinite groups.* Preliminary report.

A permutation group is k-ply pseudo-transitive if any set of k symbols can be sent to every unordered set of k symbols by permutations of the group (Carver and King, this Bulletin, vol. 26 (1920), pp. 319–322). If for any m, $0<m\leqq k$, a group of degree greater than $m+k$ has permutations which send

every unordered set of k symbols to every other which has exactly $k-m$ symbols in common with the first, the group is k-ply pseudo-transitive. If a k-ply pseudo-transitive group is infinite or of degree $k+k!$ at least, it contains a permutation which leaves invariant an arbitrary set of k symbols but moves any other given symbol. In the general case, the author has been unable to determine whether k-ple pseudo-transitivity implies pseudo-transitivity of order $k-1$ and hence of all lower orders. The theorem is true for groups that are not doubly transitive and that are infinite or of degree $k+(k-1)!(k-2)$ at least. It is also true if $k \leqq 5$ and the degree is at least $k+(k-1)!(k-2)$. For infinite groups and those of degree $k+(k-1)!(k-2)$ at least, double pseudo-transitivity is implied by pseudo-transitivity of higher order k. For finite groups, triple pseudo-transitivity implies ordinary double transitivity, 4-ple pseudo-transitivity implies triple transitivity if the degree is at least $4+3! \cdot 2=16$, and k-ple pseudo-transitivity implies double transitivity if the degree is at least $k+(k-1)!(k-2)$. (Received November 30, 1932.)

58. Dr. C. H. Harry: *Concerning spaces without local cut points.*

The main result of the present paper is the following theorem: If M is any complete, separable, metric, connected and locally connected space having no local cut point, then each compact subset K of M lies in a closed, compact, connected and locally connected subset H of M which contains no local cut point. Further, if K is connected and $\epsilon>0$ any number, then it is possible to choose H in such a manner that for each point X of H there is a point Y of K as close to X as ϵ. (Received December 1, 1932.)

59. Dr. W. T. Reid: *Note on a preceding paper.*

In a recent paper entitled *A boundary value problem associated with the calculus of variations* (American Journal of Mathematics, vol. 54 (1932), pp. 769–790) the author has given sufficient conditions for the considered differential system to have infinitely many characteristic numbers. Hypotheses (H5$^+$) and (H5$^-$) of that paper are somewhat related to the assumption of normality on every subinterval of the given interval x_1x_2. It is the purpose of the present note to point out that the existence of an infinity of characteristic numbers is obtained if these hypotheses are replaced by the following hypothesis: (H5) If η is an admissible arc which is not identically zero on x_1x_2, then the quantity $2G[\eta]+\int_{x_1}^{x_2}\eta_i(x)K_{ij}(x)\eta_j(x)dx$ is different from zero. In some respects this hypothesis is simpler than the hypotheses (H5$^+$) and (H5$^-$). It is also a condition that is satisfied by the accessory boundary value problem associated with the problem of Mayer in the calculus of variations without making the assumption of normality on every subinterval of x_1x_2. (Received November 30, 1932.)

60. Dr. W. T. Reid: *Analogues of the Jacobi condition for the problem of Mayer in the calculus of variations.*

For the general problem of Mayer in the calculus of variations, Cope (Dissertation, Chicago, 1927) has given an analogue of the necessary condition of Jacobi in terms of the characteristic numbers of a boundary value problem associated with the second variation. Bliss (Annals of Mathematics, vol. 33 (1932), pp. 261–274) has phrased the analogue of Jacobi's condition for the

problem of Bolza in terms of the characteristic numbers of a quadratic form involving only a finite number of variables, instead of in terms of the characteristic numbers of a boundary value problem of the type given by Cope. The same form of the condition has also been used by Hestenes (for abstract, see this Bulletin, vol. 38 (1932), p. 348) in treating the problem of Mayer, and the condition differs in form, therefore, from that originally given by Cope. The present paper considers the relations between these two different phrasings of the analogue of the Jacobi condition for the problem of Mayer. (Received November 30, 1932.)

61. Professor P. R. Rider: *The skewness and kurtosis of observations.*

In a paper published in 1921, Coolidge derived an expression for the expected value of the variance of n independent quantities. He also developed a general formula for the variance of a set of statistical observations which reduces for special cases to the corresponding formula for so-called Bernoulli, Lexis, and Poisson series respectively. The present paper derives formulas for the expected values of the skewness and kurtosis, and shows how the usual formulas for the third and fourth moments of Bernoulli, Lexis, and Poisson series emerge as special cases. (Received November 30, 1932.)

62. Dr. T. S. Peterson: *Linear integral equations of functions of two variables.*

It is the purpose of this paper to consider conditions for the solution of the following linear integral equation: $\bar{y}(\alpha, \beta)=y(\alpha, \beta)+\lambda\int_a^b K(\alpha, \sigma)y(\sigma, \beta)d\sigma+\mu\int_a^b L(\beta, \tau)y(\alpha, \tau)d\tau+\nu\int_a^b\int_a^b M(\alpha, \beta, \sigma, \tau)y(\sigma, \tau)d\sigma d\tau$, and, particularly, the truncated form with $M(\alpha, \beta, \sigma, \tau)\equiv 0$. For λ and μ non-characteristic values of the kernels $K(\alpha, \sigma)$ and $L(\beta, \tau)$ respectively, a necessary and sufficient condition that the above equation with $\bar{y}(\alpha, \beta)\equiv 0$, $M(\alpha, \beta, \sigma, \tau)\equiv 0$ have a solution is that λ and μ satisfy the relationship $\lambda/\bar{\lambda}+\mu/\bar{\mu}=1$, where $\bar{\lambda}$ and $\bar{\mu}$ are characteristic values of the kernels $K(\alpha, \sigma)$ and $L(\beta, \tau)$ respectively. The explicit solutions to this homogeneous equation are also determined. Certain conditions for the solution of the non-homogeneous equations of the above type are also given along with their explicit form. (Received December 2, 1932.)

63. Miss Lucile Lawrence: *On the Kasner ratio curves.*

The curves considered are a family with parameter r, $0<r<1$, each being the limit of a sequence of polygons. The first polygon is the same for all sequences, and is open, consisting of two contiguous line segments having distinct orthogonal projections on the x axis. For a given r, each polygon in a sequence is derived from the preceding by taking points on a side at one rth of the distance from each vertex to the mid-point of each adjacent side, joining these points successively in pairs, thus cutting off the corners in a definite way. It is known that for $r=1/2$, the limiting curve is a parabola. The present paper proves the following theorems: (1) Except for $r=1/2$, the third derivative fails to exist at a dense set of points, and the curve is non-analytic in any interval. (2a) For $0<r<1/2$, the second derivative, if it exists, equals zero at a dense set of points. (2b) For $1/2<r\leqq 2/3$, the second derivative, if it exists, becomes

infinite at a dense set of points. (2c) For $2/3<r<1$, the right and left first derivatives, if they exist, are unequal at each point of a dense set of points. (Received December 14, 1932.)

64. Dr. C. B. Morrey (National Research Fellow): *An analytic criterion that a surface possess finite Lebesgue area.*

A surface is said to be non-degenerate if it possesses a representation $x^i=x^i(u, v)$, $i=1, \cdots, N$, on the closed unit circle, such that the $x^i(u, v)$ are not simultaneously constant on any continuum containing two points. The representation $x^i=x^i(u, v)$ is said to be generalized conformal if (i) $x^i(u, v)$ is absolutely continuous in the sense of Tonelli, $i=1, \cdots, N$, (ii) $(x_u^i)^2$ and $(x_v^i)^2$ are summable, $i=1, \cdots, N$, and (iii) $E=G$, $F=0$ almost everywhere, where $E=\sum_{i=1}^{N}(x_u^i)^2$, $F=\sum_{i=1}^{N}x_u^i x_v^i$, $G=\sum_{i=1}^{N}(x_v^i)^2$. From recent work on the area of surfaces by McShane and the author, it is known that the Lebesgue area of surfaces thus represented is given by the usual integral formula. Thus, if we admit only non-degenerate surfaces, the following result of the present paper gives a characterization of surfaces of finite Lebesgue area: *Any non-degenerate surface, S, of finite Lebesgue area can be represented generalized conformally on the closed unit circle, R, any three logically distinct points of the boundary of S being made to correspond to three distinct points of the circumference of R.* (Received December 19, 1932.)

65. Professor A. A. Albert: *On primary normal division algebras of degree eight.*

A normal division algebra is called primary if it is not expressible as a direct product of two normal division algebras neither of degree unity. It is well known that a necessary condition that a normal division algebra be primary is that its degree be a power p^e of a prime p. Also a sufficient condition that an algebra of degree p^e be primary is that it have exponent p^e. The author has also shown that a normal division algebra of degree four is primary if and only if its exponent is four. But it has not been known until now whether primary algebras of degree greater than exponent exist. In the present paper it is proved that there exist normal division algebras that are of degree eight and exponent four, and that are primary. (Received December 19, 1932.)

66. Professor A. A. Albert: *Cyclic fields of degree eight.*

In the present paper the author constructs all cyclic fields of degree eight over any non-modular field F. The construction uses purely algebraic methods and gives as explicit formulas as are well known for the case of cyclic fields of degree four. (Received December 19, 1932.)