A PROPERTY RELATED TO COMPLETENESS*

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In 1926, R. L. Moore presented the following axiom.

AXIOM 1′. *There exists a countable sequence* $G_1, G_2, G_3, \cdots$ *such that* (a) *for each* n, G_n *is a collection of domains covering space,* (b) *if* P_1 *and* P_2 *are distinct points of a domain* R, *there exists an integer* d *such that if* $n > d$ *and* K_n *is a domain containing* P_1 *and belonging to* G_n, *then* $\overline{K}_n$ *is a subset of* $R - P_2$, *and* (c) *if* $R_1, R_2, R_3, \cdots$ *is a sequence of domains such that, for each* n, R_n *belongs to* G_n *and such that, for each* n, $R_1, R_2, \cdots, R_n$ *have a point in common, then there exists a point common to all the point sets* $\overline{R}_1, \overline{R}_2, \overline{R}_3, \cdots$.†

Moore has given an example of a non-metric space in which his axiom 1′ holds true. He raised the question as to whether or not a metric space in which his axiom 1′ holds true is *complete*.‡ The present paper answers this question in the affirmative.§

THEOREM. *A metric space* S *in which axiom* 1′ *holds true is complete.*

PROOF. Let $\delta(x, y)$ be a distance function defined over the space S. Let P be any point of S and let n be any positive integer. Either (1) there is a domain of the set G_n which contains every point y such that $\delta(P, y) \leqq 2$, or (2) there exists a greatest number k ($k \leqq 2$) such that if $r < k$, then there exists a domain of the set G_n containing every point y such that $\delta(P, y) \leqq r$. Let

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‡ A space S is said to be *complete* if there exists a definition of distance such that every sequence of points satisfying the Cauchy condition has a limit point. A sequence of points $P_1, P_2, \cdots$, in a metric space is said to satisfy the Cauchy condition with respect to the distance function δ if, for every positive number e, there exists an integer n such that $\delta(P_n, P_k) < e$ if $k > n$.

§ The present result was obtained about September 1, 1930, and was reported to Professor Moore at that time. I purposely delayed publishing the paper in order that it might not appear in advance of the publication of his book *Foundations of Point Set Theory*. Later in the fall of 1930 Leo Zippin obtained a theorem which, with *other theorems in the literature*, yields the result of this paper.

$2w$ denote 2 or k according as the first or second condition holds true. Let $C(P, n)$ denote the set of all points y such that $\delta(P, y) \leqq w$. The number w will be called the *radius of* $C(P, n)$.

NOTATION. If x, y, and Q are points, let $f(x, y; Q)$ denote the maximum of the two quantities $\delta(x, Q)$ and $\delta(y, Q)$. If n is a positive integer, let $r(Q, n)$ denote the radius of $C(Q, n)$.

Let x and y be any two points. We shall define a function $d_n(x, y)$. If there exists no point P such that $C(P, n)$ contains both x and y, then $d_n(x, y) = 1$. If there is a point P such that $C(P, n)$ contains both x and y, then let $e_{Pn}(x, y)$ be defined as $[f(x, y; P)/r(P, n)]^{1/3}$ and let $d_n(x, y)$ be the minimum or greatest lower bound of the numbers $e_{Pn}(x, y)$, where P can be any point such that $C(P, n)$ contains both x and y. A function $\rho(x, y)$ is now defined as follows:

$$(1) \qquad \rho(x, y) = \delta(x, y) + \sum_{n=1}^{\infty} d_n(x, y)/2^n.$$

It is to be shown that $\rho(x, y)$ is a distance function with respect to which S is complete. Clearly $\rho(x, y) = \rho(y, x)$ and $\rho(x, x) = 0$. If $\rho(x, y) = 0$, then $\delta(x, y) = 0$ and $x = y$. Suppose that the point x is a limit point of the point set M. Let ϵ be any positive number. Then there exists a positive integer n such that $1/2^n < \epsilon/3$. There exists a point y of M such that $\delta(x, y) < r(x, i)$ and that $[\delta(x, y)/r(x, i)]^{1/3} < \epsilon/3$ for every integer i, $(i \leqq n)$. Then since $d_i(x, y) \leqq [\delta(x, y)/r(x, i)]^{1/3}$ it follows that $d_i(x, y) < \epsilon/3$, $(i \leqq n)$. Hence

$$\rho(x, y) \leqq \delta(x, y) + \sum_{i=1}^{n} d_i(x, y)/2^i + \sum_{i=n+1}^{\infty} 1/2^i$$
$$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

Thus if x is a limit point of M, then for every positive number ϵ there is a point y of M such that $\rho(x, y) < \epsilon$. If x is not a limit point of M, then $\rho(x, M) \geqq \delta(x, M) > 0$, and there exists a positive number ϵ such that, if y is any point of M, then $\rho(x, y) > \epsilon$.

Let x, y, and z be any three points of S. It is to be shown that $\rho(x, y) + \rho(y, z) \geqq \rho(x, z)$. It is sufficient to show that for every n, $d_n(x, y) + d_n(y, z) \geqq d_n(x, z)$. Suppose that for some n we have $d_n(x, y) + d_n(y, z) < d_n(x, z)$, where x, y and z are distinct points. From the definition of the function d_n it follows that there exist points P and Q such that $C(P, n)$ contains x and y, $C(Q, n)$ contains y and z, and

$$\left[\frac{f(x, y; P)}{r(P, n)}\right]^{1/3} + \left[\frac{f(y, z; Q)}{r(Q, n)}\right]^{1/3} < d_n(x, z). \tag{2}$$

Let A denote $d_n(x, z)$. Let t and r denote, respectively, the first and second terms of the left member of (2). We next prove that

$$A^3 \leqq \left[\frac{\delta(z, P)}{r(P, n)}\right]. \tag{3}$$

This is obvious if z is not in $C(P, n)$ [that is, $\delta(z, P) \geqq r(P, n)$] since $A \leqq 1$. Let us suppose that z is in $C(P, n)$, and also that $\delta(z, P) \leqq \delta(x, P)$. Then $d_n(x, z) \leqq [\delta(x, P)/r(P, n)]^{1/3}$, which contradicts (2). Hence $\delta(z, P) > \delta(x, P)$. Then $[\delta(z, P)/r(P, n)]^{1/3}$ is one of the quantities of which $d_n(x, z)$, that is, A, is a lower bound. Hence in any case (3) is established. Similarly

$$A^3 \leqq \frac{\delta(x, Q)}{r(Q, n)}. \tag{4}$$

The following two inequalities obviously hold, since δ is a distance function:

$$\delta(z, P) \leqq \delta(z, Q) + \delta(Q, y) + \delta(y, P), \tag{5}$$

$$\delta(x, Q) \leqq \delta(x, P) + \delta(P, y) + \delta(y, Q). \tag{6}$$

From the definition of t and of r, and the fact that $r < A - t$, the following hold true, where $v_1 = r(P, n)$ and $v_2 = r(Q, n)$:

$$\begin{array}{ll} t^3 \geqq \dfrac{\delta(x, P)}{v_1}, & t^3 \geqq \dfrac{\delta(y, P)}{v_1}, \\ (A - t)^3 \geqq \dfrac{\delta(y, Q)}{v_2}, & (A - t)^3 \geqq \dfrac{\delta(z, Q)}{v_2}. \end{array} \tag{7}$$

If now (5) is divided by v_1 and substitutions are made from (3) and (7) we have

$$A^3 \leqq \frac{2(A - t)^3 v_2}{v_1} + t^3. \tag{8}$$

Likewise if (6) is divided by v_2 we have after substituting

$$A^3 \leqq \frac{2t^3 v_1}{v_2} + (A - t)^3 \cdot \tag{9}$$

Now (8) and (9) cannot both hold under the conditions here obtaining, namely $1 \geqq A > t > 0$, $v_1 > 0$, $v_2 > 0$. For from (9) we get $v_1/v_2 \geqq [A^3 - (A-t)^3]/2t^3$, from which, by (8), we find

$$3A^4 - 3A^2t^2 + 6At^3 - 3t^4 \leqq 0. \tag{10}$$

Now set $A = kt$. Then $k > 1$, and we have, after dividing by t^4, $3k^4 - 3k^2 + 6k - 3 \leqq 0$. This is obviously false. Hence our supposition has led to a contradiction, whence it follows that for every n, $d_n(x, y) + d_n(y, z) \geqq d_n(x, z)$. We can now say that the function $\rho(x, y)$ is a *distance* function.

The problem remains to show that with respect to this definition of distance the space S is complete. Let $P_1, P_2, P_3, \cdots$ denote any sequence of points satisfying the Cauchy condition with respect to the distance function $\rho(x, y)$. Let n be any positive integer and let m_n be an integer such that $\rho(P_k, P_h) < 1/2^{n+2}$ if $h, k \geqq m_n$. Then $d_n(P_h, P_k)/2^n < 1/2^{n+2}$, whence $d_n(P_h, P_k) < 1/4$ if $h, k \geqq m_n$. There exists an integer a $(a \geqq m_n)$ such that for every b, $(b \geqq m_n)$, $\delta(P_{m_n}, P_b) < 2\delta(P_{m_n}, P_a)$. Now we have $d_n(P_{m_n}, P_a) < 1/4$. Hence there exists a point Q such that

$$\left[\frac{\delta(P_{m_n}, Q)}{r(Q, n)}\right]^{1/3} < \frac{1}{4} \quad \text{and} \quad \left[\frac{\delta(P_a, Q)}{r(Q, n)}\right]^{1/3} < \frac{1}{4} \cdot$$

Then $\delta(P_{m_n}, Q) < r(Q, n)/64$, $\delta(P_a, Q) < r(Q, n)/64$, and thus $\delta(P_{m_n}, P_a) < r(Q, n)/32$. Then

$$\delta(P_b, Q) \leqq \delta(P_b, P_{m_n}) + \delta(P_{m_n}, Q) \leqq 2\delta(P_{m_n}, P_a) + \delta(P_{m_n}, Q)$$
$$< \frac{r(Q, n)}{16} + \frac{r(Q, n)}{64} < r(Q, n).$$

Hence for every b, $(b \geqq m_n)$, the point P_b belongs to $C(Q, n)$, which in turn is a subset of a domain of the set G_n. Let H_n be the set of all points P_b with $b \geqq m_n$, and let R_n be a domain of the set G_n containing H_n. By (c) of axiom 1′ there exists a point P common to all the sets $\overline{R}_1, \overline{R}_2, \overline{R}_3, \cdots$. It is easy to show that P is a sequential limit point of the sequence $P_1, P_2, P_3, \cdots$. Thus the space S is complete with respect to the distance function $\rho(x, y)$.

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