ON THE RELATIONSHIP AMONG THE DIAGONAL FILES OF A PADÉ TABLE*

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1. *Introduction.* The object of the following note is to investigate the relationship among the nth approximants of the different diagonal files of a Padé table; and to study the relationship among the limits of those files for a Stieltjes power series, in the case † that those files have different limits. We have found that an arbitrary file S_k converges to an expression of the form

$$\frac{\alpha_k p - \beta_k p_1}{\alpha_k q - \beta_k q_1}, \tag{1}$$

where p, p_1, q, q_1 are entire transcendental functions independent of k, and α_k, β_k are polynomials or constants. If we denote by u_k, v_k the numerator and denominator, respectively, of (1), then if k', k'' are two values of the index k, the following identity obtains:

$$u_{k'}v_{k''} - u_{k''}v_{k'} = \alpha_{k'}\beta_{k''} - \alpha_{k''}\beta_{k'}; \tag{2}$$

and the polynomial on the right is not identically zero if $k' \neq k''$.

2. *Preliminary Formulas.*‡ Let $\mathfrak{P}(x) = \sum_{v=0}^{\infty} c_v(-x)^v$ be a *normal* power series, and let $\mathfrak{C}(x) = \sum_{v=0}^{\infty} d_v(-x)^v$ be the reciprocal of $\mathfrak{P}(x)$. Set $\mathfrak{P}^{(k)}(x) = \sum_{v=0}^{\infty} c_{v+k}(-x)^v$, $\mathfrak{C}^{(k)}(x) = \sum_{v=0}^{\infty} d_{v+k}(-x)^v$, $k=0, 1, 2, \cdots$. Then the series $\mathfrak{P}^{(k)}(x)$, $\mathfrak{C}^{(k)}(x)$ have corresponding continued fractions

$$\frac{1}{a_1^{(k)}} + \frac{x}{a_2^{(k)}} + \frac{x}{a_3^{(k)}} + \cdots, \quad \frac{1}{b_1^{(k)}} + \frac{x}{b_2^{(k)}} + \frac{x}{b_3^{(k)}} + \cdots,$$

respectively, where the numbers $a_n^{(k)}$, $b_n^{(k)}$ are different from 0.

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† Designated as "Case I" in the writer's paper, *On the Padé approximants associated with the continued fraction and series of Stieltjes*, Transactions of this Society, vol. 31 (1929), pp. 91–116. We show in the present article that *no two* of the diagonal files have the same limit, thus supplementing the earlier result.

‡ For details concerning the statements in this paragraph, see a paper by the writer in the Transactions of this Society, vol. 33 (1931), pp. 511–532.

The Padé approximants for $\mathfrak{P}(x)$ may be expressed in terms of the numerators and denominators of the nth convergents, $A_n^{(k)}/B_n^{(k)}$, $C_n^{(k)}/D_n^{(k)}$, of these continued fractions. Indeed if $[p, q] \equiv N_{p,q}/D_{p,q}$ is that Padé approximant in which the degrees of numerator and denominator do not exceed q and p, respectively, then we have the following formulas:

$$[n-1, n+k-1] = \mathfrak{P}_k + (-x)^k A_{2n-1}^{(k)}/B_{2n-1}^{(k)}, \tag{3}$$

$$[n+k-1, n-1] = [\mathfrak{C}_k + (-x)^k C_{2n-1}^{(k)}/D_{2n-1}^{(k)}]^{-1}, \tag{4}$$

$$(n-1, k = 0, 1, 2, \cdots),$$

where $\mathfrak{P}_k = \sum_{v=0}^{k-1} c_v(-x)^v$, $\mathfrak{C}_k = \sum_{v=0}^{k-1} d_v(-x)^v$, $\mathfrak{P}_0 = \mathfrak{C}_0 = 0$;

$$[n, n-1] = A_{2n}/B_{2n}, \qquad [n-1, n] = D_{2n}/C_{2n}, \tag{5}$$

$$(n = 1, 2, 3, \cdots).$$

In the right member of (3) there occur the polynomials $A_{2n-1}^{(k)}$, $B_{2n-1}^{(k)}$. In what follows we shall want to express these polynomials in terms of the polynomials A_m, B_m. For that purpose we have the following identities which we gave in the paper to which we referred at the beginning of this paragraph, namely

$$\begin{aligned} B_{2n-1}^{(k)} &= h_n^{(k-1)} B_{2n}^{(k-1)} - B_{2n+1}^{(k-1)}, \\ xA_{2n-1}^{(k)} &= c_{k-1}(h_n^{(k-1)} B_{2n}^{(k-1)} - B_{2n+1}^{(k-1)}) - (h_n^{(k-1)} A_{2n}^{(k-1)} - A_{2n+1}^{(k-1)}), \\ h_n^{(k-1)} B_{2n}^{(k)} &= B_{2n+1}^{(k-1)}, \\ xh_n^{(k-1)} A_{2n}^{(k)} &= c_{k-1} B_{2n+1}^{(k-1)} - A_{2n+1}^{(k-1)}, \end{aligned} \tag{6}$$

$(n, k = 1, 2, 3, \cdots)$. Here $h_n^{(k-1)} = a_1^{(k-1)} + a_3^{(k-1)} + \cdots + a_{2n+1}^{(k-1)}$, and is $\neq 0$. There are four similar relations for the $C_m^{(k)}$, $D_m^{(k)}$ of (4).

3. *The Diagonal Files S_k, $k > 0$.* We shall now turn to the Padé table, which is a table of double entry containing the approximant $[p, q]$ in the $(p+1)$th row and $(q+1)$th column. The fractions A_1/B_1, A_3/B_3, A_5/B_5, $\cdots$ constitute the principal diagonal file S_0, while S_{-1}, the first parallel file below S_0, is made up of the sequence A_2/B_2, A_4/B_4, A_6/B_6, $\cdots$. If k is any integer, then the diagonal file S_k is the sequence

$$S_k: [m, m+k], \begin{cases} m = 0, 1, 2, \cdots, \text{ if } k \geqq 0, \\ m = -k, -k+1, -k+2, \cdots, \text{ if } k < 0. \end{cases}$$

The formulas (6) and (3) will enable us to express the approximants of a file S_k, $k>0$, in terms of the numerators and denominators of the fractions A_m/B_m of the files S_0 and S_{-1}. Thus if we set $k=1$ in the first two formulas of (6), we have, by (3),

$$(7)\qquad [n-1, n] = \frac{h_n A_{2n} - A_{2n+1}}{h_n B_{2n} - B_{2n+1}}\cdot$$

This is the nth approximant in the file S_1. Setting k successively equal to 1, 2 in (6), we may now express A''_{2n-1}, B''_{2n-1} in terms of the A_m, B_m. Then by (3) with $k=2$, we find that the nth approximant in the file S_2 is

$$(8)\qquad [n-1, n+1] = \frac{[(h'_n/h_n)+x]A_{2n+1} - h_n A_{2n+2}}{[(h'_n/h_n)+x]B_{2n+1} - h_n B_{2n+2}}\cdot$$

In the final step of the reduction we used the identities

$$(9)\quad A_{2n+1} = a_{2n+1}A_{2n} + xA_{2n-1},\quad B_{2n+1} = a_{2n+1}B_{2n} + xB_{2n-1}.$$

To obtain the general formula, suppose that, for a particular value of k,

$$(10)\qquad [n-1, n+k-1] = \frac{M_k A_{2n+k-1} - N_k A_{2n+k}}{M_k B_{2n+k-1} - N_k B_{2n+k}},$$

where M_k, N_k are polynomials in x in which the coefficients are rational functions of the quantities h_r^s. Let $\mathfrak{M}_k$, $\mathfrak{N}_k$ denote the polynomials obtained from these by replacing h_r^s by h_r^{s+1} throughout. Then*

$$(11)\qquad [n-1, n+k] = c_0 - x\frac{\mathfrak{M}_k A'_{2n+k-1} - \mathfrak{N}_k A'_{2n+k}}{\mathfrak{M}_k B'_{2n+k-1} - \mathfrak{N}_k B'_{2n+k}}\cdot$$

But when $k=2p$ this reduces, with the aid of (6), to (10) with $k=2p+1$, where

$$(12)\qquad M_{2p+1} = \mathfrak{M}_{2p}h_{n+p},\qquad N_{2p+1} = \mathfrak{M}_{2p} + (\mathfrak{N}_{2p}/h_{n+p});$$

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* This follows from the fact that the right member of (11) is a rational function of x in which the degrees of numerator and denominator do not exceed $n+k$ and $n-1$, respectively; and the expansion in ascending powers of x agrees with $\mathfrak{P}(x)$ for the first $2n+k-1$ terms. See Perron, *Die Lehre von den Kettenbrüchen*, Chapter X.

and when $k=2p-1$, (11) reduces to (10) with $k=2p$, where

$$(13)\quad M_{2p} = (\mathfrak{M}_{2p-1}/h_{n+p-1}) + x\mathfrak{N}_{2p-1}, \qquad N_{2p} = \mathfrak{N}_{2p-1}h_{n+p-1}.$$

By (12) and (13) we see that M_{k+1} and N_{k+1} are polynomials of the same character as M_k and N_k, namely, they are polynomials in x in which the coefficients are rational functions of the quantities h_r^s. But this has been verified for small values of k, and is therefore universally true. Formulas (12) and (13), with the initial values obtainable from (7) and (8), may be used to compute successively the polynomials M_k and N_k

4. *The Diagonal Files S_{-k}, $k>1$.* We shall next show that the approximants of the files S_{-k}, $k>1$, can be expressed in the form (10). We begin with the relations

$$\mathfrak{E}(x) \sim \frac{1}{b_1+}\,\frac{x}{b_2+}\,\frac{x}{b_3+\cdots},$$

$$\mathfrak{P}(x) = \frac{1}{\mathfrak{E}(x)} \sim b_1 - \frac{x}{(-b_2)+}\,\frac{x}{(-b_3)+}\,\frac{x}{(-b_4)+\cdots},$$

from which it follows that

$$(14)\qquad b_{2n} = -a'_{2n-1}, \quad b_{2n+1} = -a'_{2n}, \quad b_1 = c_0.$$

Also by symmetry we have

$$(15)\qquad a_{2n} = -b'_{2n-1}, \quad a_{2n+1} = -b'_{2n}, \quad a_1 = d_0.$$

Furthermore

$$c_0 - xA_n'/B_n' = D_{n+1}/C_{n+1},$$

and consequently

$$c_0B_n' - xA_n' = \gamma_nD_{n+1}, \; B_n' = \gamma_nC_{n+1},$$

where γ_n is independent of x. Giving to n the values $2p$ and $2p-1$ and employing (6) we then have

$$(16)\qquad \begin{aligned} A_{2p+1}/h_p &= \gamma_{2p}D_{2p+1}, \; B_{2p+1}/h_p = \gamma_{2p}C_{2p+1},\\ h_pA_{2p} - A_{2p+1} &= \gamma_{2p-1}D_{2p}, \; h_pB_{2p} - B_{2p+1} = \gamma_{2p-1}C_{2p}. \end{aligned}$$

For the factors γ_n we have the following values:

$$(17)\qquad \gamma_{2p} = 1, \; \gamma_{2p-1} = -1.$$

Indeed one may verify directly that the coefficients of the highest powers of x in A_{2p+1} and D_{2p+1} are 1 and $g_p=b_1+b_3+b_5+\cdots+b_{2p+1}$, respectively. Hence by (14)

$$1/h_p = \gamma_{2p}g_p = \gamma_{2p}(c_0 - a_2' - a_4' - \cdots - a_{2p}').$$

But if we take $k=1$ in the first of the following known relations*

$$(18) \qquad a_{2n}^{(k)} = a_{2n+1}^{(k-1)}/h_{n-1}^{(k-1)}h_n^{(k-1)}, \quad a_{2n+1}^{(k)} = a_{2n+2}^{(k-1)}(h_n^{(k-1)})^2, \qquad (k=1,2,\cdots;\ a_n^0 = a_n),$$

this reduces to $1/h_p=\gamma_{2p}/h_p$, so that $\gamma_{2p}=1$. In like manner we find that $\gamma_{2p-1}=-1$.

Let us now put $k=2$ in (4). Then as in (8) we find that, if we set $g_n^{(k)}=b_1^{(k)}+b_3^{(k)}+b_5^{(k)}+\cdots+b_{2n+1}^{(k)}$,

$$[n+1,\ n-1] = \frac{[(g_n'/g_n)+x]D_{2n+1} - g_nD_{2n+2}}{[(g_n'/g_n)+x]C_{2n+1} - g_nC_{2n+2}}.$$

But by (16) and (17), followed by an easy reduction in which the formulas (9), (14), and (15) play a part, this becomes

$$(19) \quad [n+1,\ n-1] = (g_n'A_{2n+1}+A_{2n+2})/(g_n'B_{2n+1}+B_{2n+2}),$$

which is the nth approximant in S_{-2}.

To obtain the general formula, we apply (10) to the series $\mathfrak{S}$, and use (4). This gives a relation of the form

$$[n+k-1,\ n-1] = \frac{V_kD_{2n+k-1} - W_kD_{2n+k}}{V_kC_{2n+k-1} - W_kC_{2n+k}},$$

where the V_k, W_k are polynomials in x with coefficients which are rational functions of the quantities g_r^s, and which may be calculated by means of formulas analogous to (12) and (13). By (16) and (17), we then have

$$(20) \qquad [n+k-1,\ n-1] = \frac{M_{-k}A_{2n+k-1} - N_{-k}A_{2n+k}}{M_{-k}B_{2n+k-1} - N_{-k}B_{2n+k}},$$

where

$$(21) \qquad \begin{aligned} M_{-2p} &= g_{n+p-1}V_{2p} - xW_{2p}, \quad N_{-2p} = -(W_{2p}/g_{n+p-1}),\\ M_{-2p-1} &= -V_{2p+1}/g_{n+p}, \quad N_{-2p-1} = g_{n+p}W_{2p+1} - V_{2p+1}. \end{aligned}$$

* Wall, Transactions of this Society, vol. 31 (1929), pp. 102–103.

The quantities M_{-k} and N_{-k} are polynomials in x with coefficients which are rational functions of the numbers g_r^s.

The recursion formulas analogous to (12) and (13) for the V_k and W_k may be combined with (21) to give the following recursion formulas for the polynomials M_{-k}, N_{-k}:

$$
(22)\qquad
\begin{aligned}
M_{-2p} &= -\,g'_{n+p-1}\mathcal{M}_{-2p+1}, &\qquad N_{-2p} &= \mathcal{M}_{-2p+1} - \frac{\mathcal{N}_{-2p+1}}{g'_{n+p-1}},\\
M_{-2p-1} &= -\frac{\mathcal{M}_{-2p}}{g'_{n+p-1}} + x\mathcal{N}_{-2p}, &\qquad N_{-2p-1} &= -\,g'_{n+p-1}\mathcal{N}_{-2p}.
\end{aligned}
$$

Here the script letters have the same significance as before, namely $\mathcal{M}_{-k}$, $\mathcal{N}_{-k}$ are polynomials obtained by replacing g_r^s by g_r^{s+1} in the corresponding polynomials M_{-k}, N_{-k}.

5. *The Series of Stieltjes.* We shall now apply the work of the preceding paragraphs to the series of Stieltjes. If we set $x=1/z$ in $\mathfrak{P}(x)$ and in the corresponding continued fraction, and then divide by z, we will obtain the series

$$
(23)\qquad \frac{c_0}{z} - \frac{c_1}{z^2} + \frac{c_2}{z^3} - \cdots,
$$

and continued fraction

$$
(24)\qquad \frac{1}{a_1z} + \frac{1}{a_2} + \frac{1}{a_3z} + \cdots.
$$

These are the forms of the series and continued fraction found in Stieltjes' *Recherches sur les fractions continues.**

The numerators and denominators, $P_n(z)$ and $Q_n(z)$, of the nth convergents of (24) are connected with the $A_n(x)$, $B_n(x)$ by the following equations:

$$
(25)\qquad
\begin{aligned}
P_{2n}(z) &= z^{n-1}A_{2n}(1/z), &\quad Q_{2n}(z) &= z^nB_{2n}(1/z),\\
P_{2n+1}(z) &= z^nA_{2n+1}(1/z), &\quad Q_{2n+1}(z) &= z^{n+1}B_{2n+1}(1/z).
\end{aligned}
$$

When $a_n>0$, and $\sum a_n$ is convergent, Stieltjes found that

$$
(26)\qquad
\begin{aligned}
\lim_n P_{2n}(z) &= p(z), &\quad \lim_n Q_{2n}(z) &= q(z),\\
\lim_n P_{2n+1}(z) &= p_1(z), &\quad \lim_n Q_{2n+1}(z) &= q_1(z),
\end{aligned}
$$

* Stieltjes, Oeuvres, vol. 2.

where $p(z)$, $p_1(z)$, $q(z)$, $q_1(z)$ are entire transcendental functions of genre 0, connected by the relation

$$p_1(z)q(z) - p(z)q_1(z) = +1. \tag{27}$$

Let us replace x by $1/z$ in (10) and (20), divide by z, and then introduce the polynomials $P_n(z)$ and $Q_n(z)$ of (25). After removing a common power of $1/z$ from numerators and denominators, these expressions then take the form

$$\frac{G_k P_{\delta-1} - H_k P_\delta}{G_k Q_{\delta-1} - H_k Q_\delta}, \tag{28}$$

where $\delta = 2n + |k|$, $(k = 0, \pm 1, \pm 2, \cdots)$. Here the G_k, H_k are polynomials in z given in terms of the M_k, N_k by the equations

$$G_k(z) = z^\theta M_k(1/z), \quad H_k = z^{\theta'} N_k(1/z), \quad \theta' = \theta - [1 - (-1)^k]/2, \tag{29}$$

where θ is the larger of the degrees of $M_k(x)$, $N_k(x)$ if k is even, and of $M_k(x)$, $xN_k(x)$ if k is odd. In particular, we have

$$G_0 = 1,\ H_0 = 0,\ G_{-1} = 1,\ H_{-1} = 0. \tag{30}$$

The others may be calculated successively by means of the recursion formulas (12), (13), (22). For example, we find that

$$\begin{aligned} &G_1 = h_n z,\ H_1 = 1;\ G_2 = (h_n'/h_n)z + 1,\ H_2 = h_n z,\\ &G_3 = (h_n'' h_{n+1}/h_n')z^2 + h_{n+1}z,\\ &H_3 = [(h_n''/h_n') + (h_n'/h_{n+1})]z + 1,\\ &G_{-2} = g_n',\ H_{-2} = -1;\ G_{-3} = -(g_n''/g_n')z - 1,\ H_{-3} = g_n'. \end{aligned} \tag{31}$$

It is seen that in these cases

$$\left.\begin{aligned} G_{2p}(0) &= 1, & H_{2p}(0) &= 0,\\ G_{-2p-2}(0) &= g_{n+p}', & H_{-2p-2}(0) &= -1,\\ G_{2p+1}(0) &= 0, & H_{2p+1}(0) &= 1,\\ G_{-2p-3}(0) &= -1, & H_{-2p-3}(0) &= g_{n+p}'. \end{aligned}\right\} \quad (p = 0, 1, 2, \cdots). \tag{32}$$

The proof of (32) for all p is readily accomplished by mathematical induction.

Now when $a_n > 0$ and $\sum a_n$ converges, we see by (18) that $a_n' > 0$ and $\sum a_n'$ converges. Applying (18) again we find that

$a_n''>0$ and $\sum a_n''$ converges, etc. Also by (15) $b_n'<0$ and $\sum b_n'$ converges. Then by (18) with $k=2$ and a_n', a_n'' replaced by b_n' and b_n'', respectively, it follows that $b_n''<0$ and $\sum b_n''$ converges, etc. Continuing in this way we find that all the series $\sum a_{2n+1}^{(k)}$, $\sum b_{2n+1}^{(k)}$ are convergent, and that their sums are >0 and <0, respectively. That is, there are finite numbers $h^{(k)}>0$, $g^{(k+1)}<0$ such that

$$(33)\quad \lim_n h_n^{(k)} = h^{(k)},\ \lim_n g_n^{(k+1)} = g^{(k+1)}, \qquad (k = 0, 1, 2, \cdots).$$

Turning now to the polynomials $G_k(z)$, $H_k(z)$ we find by (30) and (31) that for small values of k they converge, for $n=\infty$, to limiting forms which are polynomials of degree depending upon k, with positive coefficients if $k>0$, and with negative coefficients if $k<-1$. This same conclusion can be reached for all values of k by virtue of the recursion formulas (12), (13), and (22), in view of the relations (29). We shall express this result as follows:

$$(34)\quad \begin{aligned} \lim_n G_{2p} &= -\beta_{2p}, & \lim_n H_{2p} &= -\alpha_{2p},\\ \lim_n G_{2p+1} &= \alpha_{2p+1}, & \lim_n H_{2p+1} &= \beta_{2p+1}, \end{aligned}$$
$$(p = 0, \pm 1, \pm 2, \pm 3, \cdots),$$

where the $\alpha_k(z)$, $\beta_k(z)$ are polynomials in z. Furthermore by (32)

$$(35)\quad \begin{aligned} \beta_{2p}(0) &= -1, & \alpha_{2p}(0) &= 0;\\ \beta_{-2p-2}(0) &= -g', & \alpha_{-2p-2}(0) &= 1;\\ \beta_{2p+1}(0) &= 1, & \alpha_{2p+1}(0) &= 0;\\ \alpha_{-2p-3}(0) &= -1, & \beta_{-2p-3}(0) &= g', \end{aligned} \qquad (p \geqq 0).$$

Now by (34) and (26) it follows that the expression (28), which is the nth approximant in the kth diagonal file of the Padé table for the series (23), converges for $n=\infty$ to a limit u_k/v_k, where

$$u_k = \alpha_k p - \beta_k p_1,\ v_k = \alpha_k q - \beta_k q_1.$$

These functions satisfy the relation (2) as may be seen with the aid of (27). We state this result in the following theorem.

THEOREM 1. *Let* (23) *be a series of Stieltjes having a corresponding continued fraction* (24) *in which* $a_n > 0$. *Then if* $\sum a_n$ *converges, the kth diagonal file of the associated Padé table converges to the limit*

$$\frac{\alpha_k(z)p(z) - \beta_k(z)p_1(z)}{\alpha_k(z)q(z) - \beta_k(z)q_1(z)}, \tag{36}$$

where $p(z)$, $p_1(z)$, $q(z)$, $q_1(z)$ *are entire transcendental functions of* z, *and the* $\alpha_k(z)$, $\beta_k(z)$ *are polynomials in* z. *If* $u_k(z)$, $v_k(z)$ *denote the numerator and denominator, respectively, of the expression* (36), *then for two indices* k', k'' *we have the identity*

$$u_{k'}v_{k''} - u_{k''}v_{k'} = \alpha_{k'} - \alpha_{k''}\beta_{k'}. \tag{2}$$

The writer showed elsewhere that in the case under consideration, $u_{k'}/v_{k'} \not\equiv u_{k''}/v_{k''}$ if $k' \neq k''$ and $k', k'' \geqq 0$ or $k', k'' \leqq 0$. We are now able to prove the following supplementary theorem.

THEOREM 2. *Under the hypotheses of Theorem* 1, *if* $k' \neq k''$, *then* $u_{k'}/v_{k'} \not\equiv u_{k''}/v_{k''}$, *and therefore the right member of* (2) *is not identically* 0. *Furthermore, the functions* u_k/v_k *have a pole at* $z=0$ *if* $k \geqq 0$, *but are regular at* $z=0$ *if* $k<0$, *vanishing there if* $k<-1$.

In view of the earlier result mentioned above, it is clearly sufficient to prove here only the final statement in the theorem. But this follows at once from (35) and the equations*

$$p(0) = -g', \quad p_1(0) = 1, \quad q(0) = 1, \quad q_1(0) = 0.$$

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* See formulas for the $P_n(z)$, $Q_n(z)$ in Stieltjes' memoir, loc. cit. §2.