QUADRATIC PARTITIONS: PAPER IV

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1. *Identity of Degree* 5. For the preceding note, see this Bulletin, vol. 38, p. 569. The *degree* of a ϑ, ϕ identity is the degree of the identity in functions ϑ, ϕ. In II we discussed an identity of degree 4. Here we consider an identity of degree 5 whose equivalent in parity functions refers to $F(w, z, u, v)$ as in II. The identity is one of many of degree 5 by Gage.

Denote by $\Psi(w, z, u, v)$ the function

$$\phi_{111}\left(\frac{w+z}{2}, u\right)\vartheta_3^2\left(\frac{v-z}{2}\right)\vartheta_3^2\left(\frac{w-u}{2}\right).$$

Then

$$\Psi(w, z, u, v) + \Psi(w, -z, -u, -v) - \Psi(w, -z, v, u) - \Psi(w, z, -v, -u) \equiv 0$$

is an identity in w, z, u, v. The required expansions are

$$\vartheta_3(x) = \sum q^{\nu^2} \cos 2\nu x, \text{ and}$$

$$\phi_{111}(x, y) = \operatorname{ctn} x + \operatorname{ctn} y + 4 \sum q^{2n}\left[\sum \sin 2(dx + \delta y)\right].$$

For the notation in the above and in what follows, refer to I.

2. *Equivalent of Ψ-Identity.* To apply the formulas in I, §7, to the reduction of the ctn terms, make the substitution $(w, z) \rightarrow (x+y, x-y)$, and in the result apply

$$(x, y) \rightarrow ((w+z)/2, (w-z)/2).$$

Proceeding as in II, we find the following. The partitions are

$$n = 2d\delta + \nu_1^2 + \nu_2^2 + \nu_3^2 + \nu_4^2 = a_1^2 + a_2^2 + a_3^2 + a_4^2.$$

Write

$$\lambda_1 \equiv \nu_1 + \nu_2, \lambda_2 \equiv \nu_3 + \nu_4, \alpha_1 \equiv a_1 + a_2, \alpha_2 \equiv a_3 + a_4;$$

$$\sigma_1 \equiv \alpha_1 + \alpha_2, \sigma_2 \equiv \alpha_1 - \alpha_2, \sigma_{1,r} \equiv \sigma_{2,r} + 2\sigma_2,$$

$$\sigma_{2,r} \equiv 2r - 1 + e(n) - \sigma_2; S \equiv \left[\tfrac{1}{2}(|\sigma_1| - 1)\right], A \equiv \left[\tfrac{1}{2}(|\alpha_2| - 1)\right].$$

Then the identity gives

$$
\begin{aligned}
4 \sum F(d+\lambda_2, d-\lambda_1, 2\delta-\lambda_2, \lambda_1) \\
= \sum \operatorname{sgn} \sigma_1 [e(n)F(\sigma_2/2, \sigma_2/2, \alpha_1, \alpha_2) - F(\alpha_2, \alpha_1, \alpha_1, \alpha_2) \\
- 2 \sum_{r=1}^{S} F(\sigma_{2,r}/2, \sigma_{1,r}/2, \alpha_1, \alpha_2)] \\
+ \sum \operatorname{sgn} \alpha_2 [e(\alpha_2)F(\alpha_2, \alpha_1, \alpha_1, 0) + F(\alpha_2, \alpha_1, \alpha_1, \alpha_2) \\
+ 2 \sum_{r=1}^{A} F(\alpha_2, \alpha_1, \alpha_1, 2r-1+e(\alpha_2))],
\end{aligned}
$$

with F as in II.

3. *A Summation Formula*. Let $f(x_1, \cdots, x_t)$ be entire in $(x_1, \cdots, x_t)$. Then

$$\sum_{r=1}^{n} f(a_1 r + b_1, \cdots, a_t r + b_t) = f(a_1\theta(n) + b_1, \cdots, a_t\theta(n) + b_t),$$

where $\theta(n)$ is the umbra of $\theta_j(n)$ $(j=0, 1, \cdots)$ and $\theta_j(n) \equiv \sum_{r=1}^{n} r^j$. The generators of $\theta(n)$, B are

$$e^{\theta(n)x} \equiv \frac{e^x(e^{nx}-1)}{e^x-1}, \quad e^{Bx} \equiv \frac{x}{e^x-1}.$$

Hence

$$xe^{\theta(n)x} = e^{(n+1+B)x} - e^{(1+B)x},$$

and therefore, with $\theta_s(n) \equiv 0$, $s<0$,

$$r\theta_{r-1}(n) = (n+1+B)^r - (1+B)^r, \quad (r = 0, 1, \cdots).$$

4. *Application of* §3 *to* §2. Denote by $F_2(w, z, u, v)$ the function $F(w, z, u, v)$ with the restriction of entirety in (w, z, u, v). Then

$$
\begin{aligned}
4 \sum F_2(d+\lambda_2, d-\lambda_1, 2\delta-\lambda_2, \lambda_1) \\
= \sum \operatorname{sgn} \sigma_1 [e(n)F_2(\sigma_2/2, \sigma_2/2, \alpha_1, \alpha_2) - F_2(\alpha_2, \alpha_1, \alpha_1, \alpha_2) \\
- 2F_2(\theta(S) + (e(n)-1-\sigma_2)/2, \theta(S) + (e(n)-1+\sigma_2)/2, \alpha_1, \alpha_2)] \\
+ \sum \operatorname{sgn} \alpha_2 [e(\alpha_2)F_2(\alpha_2, \alpha_1, \alpha_1, 0) + F_2(\alpha_2, \alpha_1, \alpha_1, \alpha_2) \\
+ 2F_2(\alpha_2, \alpha_1, \alpha_1, 2\theta(A) + e(\alpha_2) - 1)].
\end{aligned}
$$

5. *Contractions*. We contract $F(w, z, u, v)$ with respect to z, as in II, §6. The new partitions are

$$n = \sum_{i=1}^{4} \nu_i^2 - 2\delta^2 = 2a_1^2 + a_2^2 + a_3^2 = \sum_{i=1}^{4} b_i^2,$$

restricted as next stated. Write

$$\lambda_1 \equiv \nu_1 + \nu_2,\ \lambda_2 \equiv \nu_3 + \nu_4,\ \alpha \equiv a_2 + a_3,\ A \equiv [\tfrac{1}{2}(|\alpha| - 1)],$$
$$\beta_1 \equiv b_1 + b_2,\ \beta_2 \equiv b_3 + b_4,\ \sigma_1 \equiv \beta_1 + \beta_2,\ \sigma_2 \equiv \beta_1 - \beta_2.$$

Then the restrictions are

$$\lambda_1 > 2\delta,\ e(n-1) > \sigma_2 \geqq 2 - |\sigma_1|,\ \sigma_2 \equiv e(n-1) \bmod 2.$$

With G, G_1 as in II, we have

$$\begin{aligned} &2\sum G(\lambda_1 + \lambda_2 - 2\delta,\ 2\delta - \lambda_2,\ \lambda_1 - 2\delta) \\ &= \sum \operatorname{sgn} \alpha\left[\sum_{r=1}^{A} G(\alpha, 0, 2r - e(n-1))\right] \\ &\quad + \sum \operatorname{sgn} \sigma_1 G(\sigma_2, \beta_1, \beta_2); \\ &2\sum G_1(\lambda_1 + \lambda_2 - 2\delta,\ 2\delta - \lambda_2,\ \lambda_1 - 2\delta) \\ &= \sum \operatorname{sgn} \alpha G_1(\alpha, 0, 2\theta(A - e(n-1)) \\ &\quad + \sum \operatorname{sgn} \sigma_1 G_1(\sigma_2, \beta_1, \beta_2). \end{aligned}$$

In the reductions we have used

$$1 - e(\nu) = e(\nu - 1),$$
$$2[\tfrac{1}{2}(|\nu| - 1)] = |\nu| - 1 - e(\nu),\ \sigma_1 \equiv n \bmod 2.$$

Alternative forms are obtained on noticing that

$$\operatorname{sgn} wF(w, z, u, v) = F(|w|, z, u, v).$$

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