FIVE THESES ON CALCULUS OF VARIATIONS

Contributions to the Calculus of Variations, 1930. Theses submitted to the Department of Mathematics of the University of Chicago, University of Chicago Press, July, 1931.

This volume of doctoral dissertations is of interest in the first place for its mathematical content, and secondly because it indicates a possible way of meeting the publication difficulties for theses. The University of Chicago now permits "candidates to deposit either lithoprinted or printed copies of their theses in the University library." "Owing to the moderate cost of lithoprinting extra copies it seemed an opportune time to try the experiment of distributing in collected form a limited edition of those which have been submitted during the same year or a period of years, and which are concerned with the same domain of mathematics." And so we have before us a volume of 350 pages of lithographed material, containing five theses dealing with distinct topics in the calculus of variations.

The five parts of this book treat the following topics:

1. *An envelope theorem and necessary conditions for a problem of Mayer with variable end points*, by M. G. Boyce, (pages 1–44).

2. *An historical and critical study of the fundamental lemma of the calculus of variations*, by Aline Huke, (pages 45–160).

3. *A new theory of parametric problems in the calculus of variations*, by F. L. Wren, (pages 161–194).

4. *Semi-continuity in the calculus of variations and absolute minima for isoperimetric problems*, by E. J. Mc Shane, (pages 195–244).

5. *The development of sufficient conditions in the calculus of variations*, by W. L. Duren, Jr., (pages 245–350).

In Part 1, the geometric formulation of Jacobi's necessary condition is extended to the general Mayer problem in n-space with one variable end point. The first five sections develop the multiplier-rule, and the analogues of the conditions of Weierstrass and Legendre. The general procedure is that introduced and developed by Bliss and his students; the elegance and fruitfulness of his methods become once more apparent in this work. Sections 6, 7, and 8, which contain the author's contributions to the theory, bring respectively an extension to the problem under consideration of Kneser's theorem on the envelope of a family of extremals, a geometric form of Jacobi's condition based on this theorem, and a discussion of the possibility of determining a one-parameter family of extremals which possesses an envelope with the properties needed for the application of the envelope theorem. But the difficulty which has limited the usefulness of the geometric theory in other problems again arises here; whereas it is possible to state in terms of the data of the problem conditions which insure the existence of a family of extremals with an envelope, there remains the possibility that this envelope has a singularity which interferes with its applicability. The author limits himself, since he is concerned only with necessary conditions, to a consideration of the case in which a focal point of the hypersurface on which the variable end point is movable lies between the extremities of the extremal arc E_{12}. For the study of sufficient conditions, a further consideration of the significance of the focal point would be desirable. This paper is an interesting addition to that part of the calculus of variations which deals with Jacobi's condition, and to the study of the general Mayer problem. This study has recently been materially advanced in papers by Morse (American Journal, vol. 53 (1931), p. 517) and Bliss (Annals of Mathematics, vol. 33 (1932), p. 261). It is rather a pity that the 1927 Chicago dissertation by T. F. Cope, which deals with the same problem by an entirely different method and which has not as yet been published, was not included in the present volume.

The fundamental lemma of the calculus of variations guards, like a watchdog, the entrance gates to the entire classical domain. It has to be reckoned with by any one who wishes to advance in the subject, for it is by means of the fundamental lemma, or of a modification of the fundamental lemma, that the first-order condition of a calculus of variations problem is put in the form of a differential equation. The importance of this lemma fully justifies the attention given it by successive writers, and this in turn makes desirable a systematic survey such as forms the subject of Part 2 of the present volume.

The first two chapters, covering 17 pages, give a very interesting account of the earliest work on the fundamental lemma. It furnishes a good illustration of the variability of standards of rigor in mathematical reasoning, discussed by Professor Pierpont in this Bulletin (vol. 34 (1928), p. 23). Of particular value is the recognition of the importance of the 1848 prize memoir of Sarrus. It would take some one better versed in the history of mathematics than the present reviewer to estimate the value of these chapters critically. Chapter 3 discusses modern proofs of the lemma. The rest of this part consists of a chapter on du Bois-Reymond's lemma and its extensions for the plane (24 pages), and of a final chapter on the lemma and its extensions for multiple integrals (40 pages). This is followed by a bibliography containing 131 titles. It seems to the reviewer that in an historical account of the work of other writers, absolute accuracy of statement is a prime essential. Deviation from this standard, either due to typographical errors or to a desire to abbreviate and summarize, is to be regretted. Attention is called in this connection to the statement of du Bois-Reymond's form of δy on page 71, and to the statement of Haar's theorem on page 107. The criticism on page 99 of Kryloff's proof leads me to wonder whether the author had noticed the paper by Kryloff, not listed in the bibliography, which appeared in volume 1 of the Bulletin de l'Académie de l'Ukraïne (Classe des Sciences physiques et mathématiques, p. 8), for which a list of errata has been published. The last two chapters of this part seem to the reviewer to contain a good deal of material of less value and not to reach the level of excellence of the earlier chapters. The extensions to more general integrals of Mason's lemma (Mason's paper in Mathematische Annalen, vol. 61, closes with the sentence "Der obige Beweis lässt sich ohne weiteres auf mehrere Dimensionen ausdehnen"), of Kubota's method, and of Schauder's proof of Haar's theorem are trivial generalizations. Explicit repetition on page 129 of a page-long hypothesis stated in full on page 124 is unfortunate, particularly since the only part of it which did require restatement was inadvertently omitted. It is not due to any fault of the author that the discussion of the proofs of Haar's theorem has lost a good deal of significance. In a paper published in the Hamburger Abhandlungen (vol. 8 (1930)), too late to be considered in this survey, Haar has given an exceedingly simple proof, together with a direct and valuable extension to the case of multiple integrals.

Part 3 develops the Hamilton-Jacobi theory for the integral $\int_{t_1}^{t_2}\sqrt{2f(x,x')}dt$, in which x and x' denote the sets $(x^1, \cdots, x^n)$ and $(x^{1\prime}, \cdots, x^{n\prime})$ respectively, the $'$ denoting differentiation with respect to t; the Euler equations for this problem were deduced by J. H. Taylor (this Bulletin, vol. 31 (1925), p. 257). The methods used in the present part follow in the main the procedure indicated by Bliss in his paper in vol. 17 (1916), of the Transactions, page 195, except that the author uses "orthogonal variations" η^α which satisfy the condition $f_\alpha\eta^\alpha=0$, where $f_\alpha=\partial f/\partial x^\alpha$, in place of the "normal variations" of Bliss. By this device a considerable simplification of the calculations is effected. The new material of this part is contained in the last 8 pages in which the classical Hamilton-Jacobi theory is carried through for a calculus of variations problem in this form. The author has not always used his own work to the best advantage; by means of the formula following (4.5) on page 176, Theorem VIII is proved in one line.

In the opinion of the reviewer Part 4 is the most significant of the volume. It gives most evidence of independent work and makes an important contribution to its field. It is devoted to the question as to the existence of curves (continuous and rectifiable) which furnish an absolute minimum for the integral $\int F(x, x')dt$ and give a prescribed value to the integral $\int G(x, x')dt$, in which x and x' represent respectively the set of functions $x_1(t), \cdots, x_n(t)$ and the set of their derivatives. The study is made by means of the "direct method," or "functional calculus" method, brought into the foreground in recent years through the work of Tonelli. The principal result is contained in Theorem V which gives a general set of sufficient conditions insuring the existence of an absolute minimum. As corollaries of this theorem are obtained extensions to n-space of the theorems in Tonelli's *Fondamenti* (vol. II, pp. 466, 468, 473, and 479); moreover an example is exhibited to which none of these theorems is applicable but for which the existence of an absolute minimum does follow from Theorem V.

There are several further results in this paper, among them generalizations to n-space of the theorems of Osgood and Lindeberg. The most important new tool introduced into the theory is the combination of Lemmas 1 (p. 207) and 2 (p. 208), according to which, whenever the sequence of continuous rectifiable curves $\{C_n\}$ with the limit curve C_0 is of uniformly bounded length, a subsequence $\{\overline{C}_n\}$ of $\{C_n\}$ can be so selected that, by proper choice of the parameter t, the sequences $F(\overline{C}_n)-F(C_0)$ and $\int_a^b E(x_0, x_0', \bar{x}_n')dt$ have the same superior and inferior limits, as $n\to\infty$; here $\overline{C}_n$ and C_0 are represented respectively by $x=\bar{x}_n(t)$ and $x=x_0(t)$, $a\leqq t\leqq b$, and, whenever $x_0'(t)=0$, it is to be replaced by an arbitrary unit vector such that x_0' remains measurable. In a later paper (Annals of Mathematics, (2), vol. 32 (1931), p. 580), the author has developed an analogous lemma for multiple integrals. There are some details concerning which questions might well be raised. On page 218, line 1, there is introduced a curve Σ_n obtained "by traversing and retracing portions of $\sigma(l/n)$ enough times to make $\int_{\Sigma_n} E_g(x, x', -x')dt = \int_{\Sigma_1} E_g(x, x', -x')dt$." Does such a curve Σ_n always exist? In the discussion on pages 222 and 223, it would facilitate the reading considerably if it had been made clear why $h_0 \geqq H(C_0)$.

Part 5 is an historical account of the development of sufficient conditions for various problems of the calculus of variations. The survey is less critical in character than the one which constitutes Part 2; it consists more largely of a restatement of results obtained by different authors, and hence it seems to the reviewer less significant. Nevertheless it is of some interest to have the important problem of sufficiency traced systematically through the history of the subject. There are a few instances in which the reviewer would have liked to find greater accuracy of statement in the report of earlier work. The lemma attributed to Levi on page 276 is not an exact statement of the one used by that author; the multiplier rule, for which reference is given to Bliss, 1925 Chicago lectures (since then printed in American Journal, vol. 52 (1930), pp. 674–743) is put in a less general form than that given it by Bliss. It is not made clear what the exact relation is between the normality hypothesis of Bolza and that of Bliss in the sufficiency theorem stated on page 310. While the restatement in toto of the theorem of Bliss on page 308 is perhaps justifiable on the ground that the 1925 Chicago lectures were not generally available at the time this paper was written, no justification seems to exist for reproducing in full on pages 316–318 the theorem of Lindeberg of which a complete account is to be found on pages 515–519 of Bolza's *Vorlesungen*.

The various sufficiency proofs for a particular type of problem are classified according to the method of proof. After an introductory Chapter (pp. 1–14), come Chapter II on expansion methods (pp. 15–27), Chapter III on Weierstrass' field method for simple problems (pp. 28–38), and Chapter IV on methods related to the field method (pp. 39–46). The five remaining chapters deal with "more general problems," such as the problems of Lagrange and Mayer (pp. 47–63), isoperimetric problems (pp. 64–68), problems with variable end points (pp. 69–77), "problems of special types," broken extremals, one-sided variations and closed extremals (pp. 78–84), and multiple integrals (pp. 85–90). A bibliography of 121 titles closes this part. The rapid progress made the last few years in the study of sufficient conditions has made this survey incomplete even so short a time after its publication. The papers of Morse, American Journal, vol. 53, and of Bliss, Annals, vol. 33, referred to above, have served to fill the gap mentioned at the bottom of page 298. They were published after this book had been completed. On the other hand Part 4 of the book contains results which might well have been taken into account in Part 5.

This brings us to a consideration of the volume as a whole. It is perhaps a result of the circumstances which gave rise to its publication that there is no attempt to connect its different parts. Whatever unity it possesses comes from the fact that the five different parts are all concerned with the calculus of variations. Would not a good deal be gained if, in future volumes of this sort, an additional chapter were included in which the relations between the different parts, and perhaps supplementary material as well, could be considered? Some editorial unification of the volume would also be desirable. Apart from the statement that the theses in this volume "were written under the direction of Professors G. A. Bliss and L. M. Graves", there is no indication of editorial supervision. Neither the title-page nor the unsigned preface suggest that any one has assumed responsibility for anything except his own part of the book. As might be expected under such circumstances, the proof-reading has been unevenly done. While the straight text is lithoprinted from typewritten material, all formulas and symbols not usually found on a typewriter-keyboard have been done from material written in by hand. This material shows great unevenness in legibility and in completeness. In a good many places symbols which should have been filled in have been omitted. It is true that the reviewer did not notice any misleading misprints or omissions.

It would seem that the principal gain to be derived from publishing a number of such theses jointly would lie in the attainment of a certain unity among them and in securing for them some editorial service. Should this not be feasible, the publication of the thesis in lithoprinted form could better be undertaken by each candidate individually.

A final question is in the reviewer's mind. Should doctors' theses be subjected to such critique outside the bosom of the Alma Mater which nurtured their authors as it is desirable to exercise on the publications of authors who stand entirely on their own responsibility? This is primarily a question to be considered by the Editors of the Bulletin. Had they answered it by *No*, the reviewer would have been saved a good deal of labor. It is partly because of the belief that they would answer the question affirmatively that the present review was written.

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