ON SURFACES IN SPACE OF r DIMENSIONS

BY B. C. WONG

Consider a surface F^n of order n in r-space. Let it be the complete intersection of $q \leqq r-2$ varieties $V_{k_1}^{n_1}, V_{k_2}^{n_2}, \cdots, V_{k_q}^{n_q}$ of orders $n_1, n_2, \cdots, n_q$ and of dimensions $k_1, k_2, \cdots, k_q$, respectively, where

$$\text{(A)} \qquad \begin{aligned} 3 \leqq k_1, k_2, \cdots, k_q \leqq r-2, \\ k_1 + k_2 + \cdots + k_q = r(q-1) + 2. \end{aligned}$$

Project F^n on an S_3. The projection F'^n has a number of characteristics of which we note the following six: n, its order; a, the order of its tangent cone; b, the order of its double curve; j, the number of its pinch-points; t, the number of its triple points; and m, its class. If we project F^n on an S_4, the projection has a finite number, d, of improper double points. We shall call these seven characteristics, of which n, a, t, m are often regarded as essential, the characteristics of F^n, and they are known to satisfy the following relations:*

$$\text{(B)} \qquad \begin{aligned} a + 2b = n(n-1), \quad j + 2d = n(n-1) - a, \\ j = \tfrac{1}{4}[a(3n-4) - n(n-1)(n-2) + 6t - 2m], \\ d = \tfrac{1}{8}[n(n-1)(n+2) - 3an - 6t + 2m]. \end{aligned}$$

For $r=5$, $q=3$, $k_1=k_2=k_3=4$, F^n is the intersection of three hypersurfaces in S_5. Formulas for its characteristics are known† and they are symmetric functions of the orders of the hypersurfaces. In this note we present analogous formulas for the same characteristics of F^n for r general and for $q \leqq r-2$. As the method of obtaining these formulas is familiar and has been applied by the writer time and again to similar enumerative problems,‡ we shall here omit all demonstration.

---

* Severi, *Intorno ai punti doppi impropri di una superficie generale dello spazio a quattro dimensioni, e a'suoi punti tripli apparenti*, Rendiconti di Palermo, vol. 15 (1901), pp. 33–51.

† B. C. Wong, *On surfaces in spaces of four and five dimensions*, this Bulletin, vol. 36 (1930), pp. 681–686. Opportunity is here taken to correct an error in the formula for T'' on page 685 of this paper. The formula should read

$$T'' = \tfrac{1}{2}\lambda\mu\nu(\lambda-1)(\mu-1)(\nu-1)(\mu\nu+\nu\lambda+\lambda\mu-2\lambda-2\mu-2\nu).$$

‡ B. C. Wong, loc. cit., and also the paper *On the number of apparent double points of r-space curves*, this Bulletin, vol. 37 (1931), pp. 421–423.

If $q=r-2$, and, from (A), $k_1=k_2=\cdots=k_{r-2}=r-1$, F^n is the complete intersection of $r-2$ hypersurfaces in S_r. The formulas for its characteristics are

$$\begin{aligned}
n &= n_1n_2\cdots n_{r-2},\\
a &= n(\sum n_i - r + 2),\\
b &= \tfrac{1}{2}n(n - \sum n_i + r - 3),\\
d &= \tfrac{1}{2}n[n - \sum n_in_j + (r-4)\sum n_i - \tfrac{1}{2}(r-3)(r-4)],\\
j &= n[\sum n_in_j - (r-3)\sum n_i + \tfrac{1}{2}(r-2)(r-3)],\\
t &= \tfrac{1}{6}n[n(n - 3\sum n_i) + 3(r-3)(n - 2\sum n_i)\\
&\quad + 2(\sum n_i^2 + 3\sum n_in_j) + (r-3)(3r-8)],\\
m &= n[\sum(n_i-1)^2 + \sum(n_i-1)(n_j-1)], \qquad (i\neq j).
\end{aligned}$$

Now if $q\leqq r-2$, one or more of the k's will be less than $r-1$. Let the ith variety $V_{k_i}^{n_i}$ be intersected by a general S_{r+2-k_i} in a surface F^{n_i}. We assume known the characteristics a_i, b_i, t_i of F^{n_i} besides n_i. The characteristics of F^n are given by the following formulas which are functions of n_i and q, and also of a_i, b_i and t_i:

$$\begin{aligned}
n &= n_1n_2\cdots n_q,\\
a &= n(\sum n_i - q) - 2n\sum b_i/n_i = n\sum a_i/n_i,\\
b &= \tfrac{1}{2}n(n - \sum n_i + q - 1) + \sum b_i/n_i = \tfrac{1}{2}n(n-1) - \tfrac{1}{2}n\sum a_i/n_i,\\
d &= \tfrac{1}{2}n[n - \sum n_in_j + (q-2)\sum n_i - \tfrac{1}{2}(q-1)(q-2)]\\
&\quad + n\sum_i(\sum n_j - q + 1)b_i/n_i - 2n\sum b_ib_j/n_in_j,\\
j &= n[\sum n_in_j - (q-1)\sum n_i + \tfrac{1}{2}q(q-1)]\\
&\quad - 2n\sum_i(\sum n_j - q)b_i/n_i + 4n\sum b_ib_j/n_in_j,\\
t &= \tfrac{1}{6}n[n(n - 3\sum n_i) + 3(q-1)(n - 2\sum n_i)\\
&\quad + 2(\sum n_i^2 + 3\sum n_in_j) + (q-1)(3q-2)] + n\sum t_i/n_i\\
&\quad + n\sum_i[n - 2\sum n_j - n_i + 2(q-1)]b_i/n_i + 4n\sum b_ib_j/n_in_j,\\
m &= n[\sum(n_i-1)^2 + \sum(n_i-1)(n_j-1)] + 3n\sum t_i/n_i\\
&\quad - n\sum_i(2\sum n_j + 3n_i - 2q + 2)b_i/n_i + 4\sum b_ib_j/n_in_j,\\
&\qquad\qquad (i\neq j).
\end{aligned}$$

All the formulas of each of these two sets satisfy relations (B).

UNIVERSITY OF CALIFORNIA