REPRESENTATION OF A GROUP AS A TRANSITIVE PERMUTATION GROUP

BY G. A. MILLER

Let G be any group of finite order g and let H be any subgroup of order h contained in G. If the operators of G are separated into right or into left augmented co-sets with respect to H and if these $g/h=n$ co-sets are then multiplied successively on the right or on the left respectively by the various operators of G, they will be permuted as units according to a transitive permutation group T which is simply isomorphic with the quotient group of G with respect to the largest invariant subgroup of G which appears in H, if H is not itself invariant under G. If H is invariant under G, then T will be a regular group which is simply isomorphic with G/H. The case when H is non-invariant under G and does not involve any invariant subgroup of G besides the identity is especially important since T is then simply isomorphic with G, as was pointed out for right co-sets by W. Dyck in 1883.

If K is any subgroup of G which has operators in each of the co-sets of G with respect to H and if K_0 is the cross-cut of H and K, then K_0 may be invariant under K or it may involve an invariant subgroup under K. If one of these conditions is satisfied, H must involve a subgroup which is invariant under G and includes this invariant subgroup under K. This follows directly from the facts that this invariant subgroup is transformed into all of its conjugates under G by operators of H and that a complete set of conjugate subgroups always generates an invariant subgroup if it does not generate the entire group. We have then the following result.

THEOREM 1. *If a group G is separated into co-sets with respect to a subgroup H and if another subgroup K has operators in each of these co-sets, then the largest invariant subgroup under K which appears in the cross-cut of H and K is contained in an invariant subgroup of G which is found in H.*

In particular, when T is simply isomorphic with G, then the largest invariant subgroup under K which appears in the cross-cut of H and K is the identity.

A necessary and sufficient condition that a subgroup of G corresponds to a transitive subgroup of degree n in T is that it has operators in each of the co-sets of G with respect to H. In the case when T is simply isomorphic with G this transitive subgroup of degree n must also be simply isomorphic with the corresponding subgroup of G. We have then the following result.

THEOREM 2. *When G is represented as a simply isomorphic transitive permutation group of degree n with respect to a subgroup H, then a necessary and sufficient condition that a given subgroup of G corresponds to a simply isomorphic transitive subgroup of degree n is that operators of this subgroup of G appear in each of the co-sets of G with respect to H.*

When T is not simply isomorphic with G, a subgroup of G is not necessarily simply isomorphic with the corresponding subgroup of T. A necessary and sufficient condition that such a simple isomorphism exists is that this subgroup of G has only the identity in common with the largest invariant subgroup of G which is found in H. When H is invariant under G, then the regular group T will correspond to every subgroup of G whose operators are distributed among all of the co-sets of G with respect to H. Subgroups of G whose operators are not thus distributed will correspond to regular constituents of the subgroups of T. A necessary and sufficient condition that a subgroup of G corresponds either to a regular subgroup of T or to a regular constituent of a subgroup of T is that its cross-cut with H is invariant under this subgroup.

Suppose that G involves a subgroup H whose operators are distributed among all except one of the co-sets of G with respect to H and that T is simply isomorphic with G. The subgroup H must correspond to a simply isomorphic subgroup of T which is of degree $n-1$ and transitive on these $n-1$ letters. This subgroup must therefore appear in a conjugate of the subgroup of T which corresponds to H. It therefore results that T is multiply transitive. Moreover, when T is multiply transitive, G must contain such a subgroup.

THEOREM 3. *A necessary and sufficient condition that a group G appears as a multiply transitive group when it is represented as a transitive permutation group with respect to a subgroup H is that at least one subgroup of G has its operators distributed among all except one of the co-sets of G with respect to H.*

From this theorem it results directly that a necessary and sufficient condition that a group is r-fold transitive when it is represented as a simply isomorphic transitive group with respect to a subgroup H is that one can find $r-1$ successive subgroups $H_1, H_2, \cdots, H_{r-1}$ each of which after the first is contained in the preceding and has its operators distributed among all except one of the co-sets of G with respect to H in which the operators of the preceding subgroup are found, the operators of H_1 appearing in all these co-sets except one. It may be noted that the operators of all of these H's have the property that their products into any co-set which contains no operator of the corresponding H appear in this co-set.

Suppose that all the operators of G are distributed successively with respect to a subgroup H of G both into right and also into left co-sets. It is easy to verify that every such right co-set is identical with some left co-set of G with respect to a conjugate of H and that the totality of the right co-sets of H includes left co-sets of G with respect to all the conjugates of H under G. Similarly, the totality of the left co-sets with respect to H includes right co-sets with respect to all the conjugates of H under G. This results directly from the theorem that the multiplying operators of co-sets can always be so chosen that the totality of the right multipliers is identical with the totality of left multipliers. This proves the following theorem.

THEOREM 4. *The right co-sets of any group G with respect to a given subgroup H are composed of left co-sets of G with respect to all the conjugates of H under G, and vice versa.*

If more than one right co-set of G with respect to H is equal to a left co-set with respect to the same conjugate of H, the number of such right co-sets is equal to the index of H under the largest subgroup of G in which H is invariant. In particular, this number is an invariant of G. A necessary and sufficient condition that only one right co-set of G with respect to H is equal to a left co-set with respect to a given conjugate of H is that H is transformed into itself only by its own operators under G. This is also a necessary and sufficient condition that only one left co-set of G with respect to H is equal to a right co-set with respect to a given conjugate of H. The operators of G which when multiplied on the right into a given right co-set of G with respect to H have all their products in this co-set constitute the conjugate of H in the equivalent left co-set, and vice versa.

A necessary and sufficient condition that T is simply transitive is that at least one conjugate of H under G has its operators distributed among less than $n-1$ co-sets of G with respect to H. In particular, when these operators are distributed among $n-2$ such co-sets n must be even and T must involve a system of imprimitivity composed of $n/2$ sets of letters. This is also a necessary and sufficient condition that H is invariant under a subgroup of G whose order is exactly $2h$. If G is simply isomorphic with T and a subgroup of G has its operators distributed among $n-2$ of the co-sets of G with respect to H, its order cannot exceed $2h$, and when it has this order it must involve a subgroup of index 2 which is conjugate with H under G. Moreover, H corresponds to a transitive subgroup of degree $n-2$ in T.

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A NOTE ON TRANSFINITE ORDINALS

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In a supplementary note to an article of theirs,* Alexandroff and Urysohn demonstrated the following theorem.

If to every ordinal α of the second class there corresponds an ordinal $\mu(\alpha)$ such that $\mu(\alpha)<\alpha$, then there exists a non-denumerable set of ordinals of the second class

$$\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots, \alpha_\omega, \cdots, \alpha_\lambda, \cdots$$

such that

$$\mu(\alpha_1) = \mu(\alpha_2) = \cdots = \mu(\alpha_\lambda) \cdots .$$

The present note applies a different method to prove the following more general result.

THEOREM. *Let Ω_δ be the smallest ordinal whose power is $\aleph_\delta$, where $\delta>0$ is a non-limiting ordinal. If to every transfinite ordinal $\alpha<\Omega_\delta$ there corresponds an ordinal $\mu(\alpha)$ such that $\mu(\alpha)<\alpha$,*

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* *Mémoire sur les espaces topologiques compacts*, Verhandelingen of the Amsterdam Academy, (1), vol. 45, No. 1.