DUNHAM JACKSON ON APPROXIMATION

The Theory of Approximation. By Dunham Jackson. New York (American Mathematical Society Colloquium Publications, Volume 11), published by this Society, 1930. v+178 pp.

In 1885 Weierstrass announced his now famous theorem: *Any continuous function $f(x)$, defined over a finite interval (a, b), can be approximated in (a, b) uniformly and indefinitely by a sequence of polynomials $P_n(x)$, whose degrees n tend to infinity,* that is, $\lim_{n\to\infty} P_n(x)-f(x)=0$, $(a\leqq x\leqq b)$. The same holds true, if polynomials are replaced by finite trigonometric sums of ever increasing orders. We shall speak of this, for brevity, as the (ordinary) polynomial or trigonometric approximation.

One can hardly overestimate the influence exerted by this theorem, now commonly known as "Weierstrass' Theorem," on the development of modern analysis. It suffices to say that the theorem in question very often enables the investigator to extend, at a single stroke, a property previously established *for polynomials only* to the infinitely wider class of continuous functions.

It seems, therefore, quite fitting that the *Theory of Approximation* is the subject of one of the series of the Colloquium Lectures of the Society, a subject for which no one is better qualified as lecturer than Professor Dunham Jackson, who has made to it such important contributions. Those who had the pleasure of hearing the Colloquium Lectures or other papers read before the Society by Professor Jackson always admire their lucidity and elegance of style. The present book possesses these qualities to a high degree. The exposition is clear, detailed and interesting, and in many points distinctly novel.

The book consists of five chapters. Chapter 1 is devoted to the ordinary trigonometric and polynomial approximation of continuous functions. After a brief introduction, where Weierstrass' Theorem is stated, the author proves the fundamental Theorem 1 on trigonometric approximation of a function $f(x)$ satisfying a Lipschitz condition. The proof is based upon the use of the *singular* integral

$$h_m \int_{-\pi/2}^{\pi/2} f(x+2u)F_m(u)du, \quad F_m(u) = \left(\frac{\sin mu}{m \sin u}\right)^4, \quad \frac{1}{h_m} = \int_{-\pi/2}^{\pi/2} F_m(u)du,$$

introduced by Jackson in his Thesis; the analysis is carried out in a detailed manner. This leads to Theorem 2 on trigonometric approximation of any continuous function, given its modulus of continuity $\omega(\delta)$, through an ingenious device also introduced by the author in his Thesis. A closer analysis of the approximating sum employed in Theorem 1 leads further to Theorems 3, 4 on trigonometric approximation of a function whose pth derivative satisfies a Lipschitz condition or has a given modulus of continuity. In the next section the author states similar theorems for polynomial approximation, reducing this to the previous case by means of the substitution $x=\cos\theta$. The next section is devoted to a brief discussion of the rapidity of convergence of Fourier and Legendre series, which is taken up with more details in Chapter 2. We note an extremely elegant proof (pages 27–28) of the following property of Legendre's polynomials $X_k(x)$ (of degree k): $|X_k(x)|\leqq c/(k^{1/2}(1-x^2)^{1/2})$, $(-1<x<1$; $c=\text{const.})$. The chapter closes with a brief remark on approximation by means of more general classes of functions.

Chapter 2 deals with the approximation of discontinuous functions, especially of those of bounded variation. The central theme is Fourier series. The author uses the Lebesgue theory of integration and discusses the convergence of Fourier series, first, for summable functions satisfying in a part of the period certain conditions of continuity. Due to the general character of the functions under consideration, the degree of convergence is not discussed, but we find a clear and interesting discussion of it in the following section dealing with functions of bounded variation or whose derivative of a certain order is of bounded variation. Next comes the convergence and the degree of convergence of the first arithmetic mean, *Fejér sum*, of the Fourier series for a given summable function, and its degree of convergence for functions satisfying certain conditions of continuity. Here (page 63) the author makes the interesting remark that no higher order of approximation of $f(x)$ by means of its Fejér sum would be obtained by supposing $f(x)$ provided with additional derivatives (besides $f'(x)$), using as illustration the analytic function $\cos x$. A similar discussion is further carried out for Legendre series. Here (page 73) we notice again an elegant estimate of $|X_{n-1}(x)-X_{n+1}(x)|$ by means of the Laplace integral used in Chapter 1.

Chapter 3 is devoted to trigonometric and polynomial approximation based on the principle of least squares or, more generally, of least mth powers, that is, the approximation of $f(x)$ by $\phi(x)$ on (a, b) being measured by

$$\int_a^b \rho(x)\,|f(x)-\phi(x)|^m dx, \qquad (m>0;\ \text{weight-function } \rho(x)\geqq 0 \text{ on } (a,b)).$$

The minimum property involved in the least squares principle is established, first, for any set $p_0(x)$, $p_1(x)$, $\cdots$ of normalized orthogonal functions over a given interval (a, b), and later applied to the set of trigonometric functions $\{\sin mx, \cos mx\}$, $(m=0, 1, \cdots)$, where the least squares principle leads to the Fourier expansion of the function we seek to approximate. Sufficient conditions for the convergence of the minimizing sum to $f(x)$, under proper conditions of continuity, are then established in a simple and elegant manner, by means of the well known Bernstein Theorem: $\max|T_n(x)|\leqq L$ implies $\max|T_n'(x)|\leqq nL$, ($T_n(x)$ a trigonometric sum of order n). There follow various generalizations: introduction of a weight-function, together with the least mth powers. The results obtained are then extended to the case of polynomial approximation, by making use of the generalized Bernstein Theorem and its various corollaries as applied to polynomials. Especially noteworthy is Corollary III (page 94):

$$|P_n(x_2)-P_n(x_1)|\leqq \frac{4nL}{(b-a)^{1/2}}|x_2-x_1|^{1/2}, \text{ if } |P_n(x)|\leqq L \text{ on } (a, b),$$

$$(a\leqq x_1, x_2\leqq b;\ P_n(x) \text{ a polynomial of degree } n).$$

Legendre's series here again presents itself as the simplest case ($\rho(x)\equiv 1$, $m=2$). The chapter ends with the least squares polynomial approximation and its convergence over the infinite interval $(-\infty, \infty)$, with the weight-function e^{-x^2}, that is, using Tchebycheff-Hermite polynomials. The author justly remarks in closing that the method employed yields little, if anything new, being of an elementary nature (it does not make use, for instance, of the asymptotic expression for a Tchebycheff-Hermite polynomial of a very high degree).

Chapter 4 is devoted to trigonometric interpolation, intending to show, as the author says at the beginning, "certain striking analogies · · · between the theory of interpolation by means of trigonometric sums and that of Fourier series." An interpolation formula is given with equidistant abscissas, which, regarding its coefficients and convergence properties, bears a close resemblance to the partial sum of the Fourier expansion. The degree of its convergence is discussed for functions continuous over the entire period or over a part of it, or of bounded variation. The discussion follows the same lines as used previously (Chapters 1 and 2), and the results are very similar, thus confirming the passage quoted above. In the second part of this chapter the author derives an interpolation formula (introduced by himself in analysis) which possesses the remarkable property that *it converges uniformly for any continuous periodic function* (to the value of the function in question), in common with the Fejér sum discussed in Chapter 2. Its degree of convergence is discussed for functions satisfying a Lipschitz condition.

Chapter 5 deals with the fundamental notions of geometry of function space. We find here, first, the least squares approximation principle applied to "polynomials" $\sum_{k=0} c_k p_k(x)$, (c_k=const.), $\{p_k(x)\}$ representing a general set of linearly independent functions defined over (a, b). The author proceeds with the geometric interpretation in function space of distance, orthogonality, and, more generally, of the notion of an angle. This yields an interesting interpretation of coefficients of correlation. The chapter ends with geometrical considerations briefly applied to frequency functions and to vector analysis in function space.

In any given book one can find shortcomings and omissions which very often represent a function of the mental and artistic tastes of the reviewer Such, perhaps, is the case with the remarks which follow.

We find missing some commonly used terms which, in some instances, could have shortened subsequent statements. Such is the case with the term "Lipschitz condition": $|f(x_2)-f(x_1)| \leqq \lambda |x_2-x_1|$, introduced on page 2 and used extensively throughout the book (later, page 22, the author uses the term "Lipschitz-Dini condition" for $\lim_{\delta \to 0} \omega(\delta) \log \delta = 0$). The same must be said of the term "Fejér sum" for the first arithmetic mean of partial sums of Fourier series (page 57) (later the author refers to it as "Fejér mean"), of the term "Laplace integral" for the integral $(1/\pi)\int_0^\pi [x+i(1-x^2)^{1/2}\cos\phi]^k dx$ (page 26), also of the term "Schmidt's orthogonalization process" (page 90).

Some proofs could be abbreviated, as, for example, that on page 27, or on pages 78–79, where the author proves that, given a sequence $P_0(x), P_1(x), \cdots$ of normalized orthogonal functions, summable with their squares on (a, b), min $\int_a^b [f(x)-\sum_{k=0}^n c_k P_k(x)]^2 dx$ is attained, if and only if $c_k = \int_a^b f(x)P_k(x)dx$. It suffices to consider, we believe, the expression

$$\int_a^b \left[f(x) - \sum_{k=0}^n (h_k + c_k)P_k(x)\right]^2 dx$$

$$= \int_a^b f^2(x)dx - 2\sum_{k=0}^n (c_k + h_k)c_k + \sum_{k=0}^n (c_k + h_k)^2$$

$$= \int_a^b f^2(x)dx - \sum_{k=0}^n c_k^2 + \sum_{k=0}^n h_k^2$$
$$\geqq \int_a^b f^2(x)dx - \sum_{k=0}^n c_k^2,$$

where c_k is as stated above, and h_k is an arbitrary constant; and the statement given above follows directly.

On the other hand, on page 69 it seems advisable to state *fully* Schwartz's inequality, which still has not (although it should have) found its way into our textbooks. Also the statement concerning the limit of the approximation of an arbitrary continuous function (page 8) should be stated more clearly (as in the author's thesis): $\psi(n)$ being an arbitrary positive non-increasing function such that $\lim_{n\to\infty}\psi(n)=0$, there exists a function $f(x)$, continuous in (a, b), for which $\psi_f(n)\geqq\psi(n)$, $\psi_f(n)=$the best approximation of $f(x)$ in (a, b) by means of a polynomial of degree n. Also the process of orthogonalizing a given sequence of functions $\{(q_n(x)\}$ (page 90) can be made clearer and more easily to be memorized by the simple remark that each new orthogonalized $Q_k(x)$ is the remainder in the formal development of $q_k(x)$ in a series of previously orthogonalized $Q_0(x), Q_1(x), \cdots, Q_{k-1}(x)$.

May we add a few more remarks. At the beginning the reviewer would like to see introduced, if briefly, the notion of "best approximation," in the Tchebycheff sense. It plays such an important role in the theory of approximation and could be used also, when discussing the degree of convergence of Fourier and Legendre series, in order to give the reader an idea how favorably the approximation in question compares with the best possible one.

When introducing the integral $h_m\int_{-\pi/2}^{\pi/2} f(x+2u)\,[\sin mu/(m\sin u)]^4du$ (page 3), one would like to see a reference to *singular integrals* in general, stating the characteristic properties of the integrand to which the success of their application is due. Similarly, when discussing Legendre series, one might mention, in connection with the order of magnitude of $\int_{-1}^{1}|\sum_n(x, t)|dt$ (pp. 31–32), the so-called "Lebesgue constants" and their role in the theory of series of orthogonal functions in general. Furthermore, the close similarity of the results obtained for Legendre series to those established for Fourier series makes very desirable a remark to the effect that this similarity is essentially due to the fact that for $-1<x<1$ the normalized Legendre polynomial $(2k+1/2)^{1/2}X_k$ behaves, for k very large, like $\cos kx$ or $\sin kx$. (In fact, we know at present how closely related are, with regard to convergence and divergence, these two series.)

When dealing with approximation based on the least mth powers principle, one wishes to see mentioned the result of Pólya (Comptes Rendus, vol. 157 (1913)) establishing a remarkable relation between this method of approximation and the (best) ordinary one.

Finally, when discussing the trigonometric interpolation formula analogous to Fejér's mean (pp. 142–148), the author makes a parenthetical reference to an article by Fejér. We believe this reference should be amplified by citing from that article Fejér's interpolation formula, which has the same important property of being convergent for any continuous function.

The reviewer believes that remarks of the kind mentioned above, even made in footnote form, tend to enlarge the horizon of the reader and to stimulate his mental curiosity.

The aforesaid critical remarks do not, of course, detract from the merit of the book under review. The reader will enjoy it as interesting, stimulating, extremely well written, and beautifully printed. Many a worker will be inspired by the pages devoted to Fourier and Legendre series and to Tchebycheff-Hermite polynomials to try his hand in obtaining similar results for other classes of orthogonal functions, and in particular, for other classes of orthogonal Tchebycheff polynomials.

As to the choice of material, the reviewer would prefer to see the last chapter devoted not to geometry of function space, but to approximation over the real axis of functions with singularities. However, in the preface the author expresses himself as follows: "The title of this volume is an abbreviation for the more properly descriptive one: *Topics in the Theory of Approximation*. It is an account of certain aspects and ramifications of a problem to which I was introduced at an early stage, and which has given direction to my reading and study ever since." This is very disarming indeed. The author is certainly the supreme judge in choosing his material.

May we, therefore, express the hope which, we feel certain, is shared by all readers of this book, that it will be soon followed by a second one devoted to the same important field, written, needless to say, with the same mastery of style and scholarly authority.

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