ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THIS SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

172. Professor R. L. Jeffery: *Volterra integral equations of the first kind.*

This paper deals with a study of the equation (1) $f(x)=\int_0^x K(x,\xi)u(\xi)d\xi$ under the fgllowing conditions: (a) $f(0)=0$, $K(x,\xi)$ integrable in ξ for each x, $\{f(x)-f(x')\}/(x-x')$ and $\{K(x,\xi)-K(x',\xi)\}/(x-x')$ bounded; (b) the left hand derivatives $f'^{-}(x)$ and $K_x^{-}(x,\xi)$ respectively of $f(x)$, and of $K(x,\xi)$ with respect to x, existing; (c) conditions on $\phi(x,x')=\{\int_{x'}^{x}K(x,\xi)d\xi\}/(x-x')$. If (a) holds and $\phi(x,x')$ is bounded uniformly from zero for x' sufficiently close to x, then functions $u_\epsilon(x)$ such that $|f(x)-\int_0^x K(x,\xi)u_\epsilon(\xi)d\xi|<\epsilon$ exist for any given ϵ. If (a) and (b) hold, and if $\lim_{x'\to x}\phi(x,x')=\rho(x)\times 0$, then (1) has a unique bounded solution given by a series involving $f'^{-}(x)$, $K_x^{-}(x,\xi)$, and $\rho(x)$. If $\rho(x)\equiv 0$, let $\rho_1(x)$ be obtained by replacing $K(x,\xi)$ with $K_x(x,\xi)$ in the limiting process which gives $\rho(x)$. If then $\rho_1(x)\neq 0$, $f''^{-}(x)$ and $K_{xx}^{-}(x,\xi)$ both exist, there is no solution of (1) unless $f(0)=f'(0)=0$. If this last condition holds, equation (1) has a unique bounded solution given by a series involving $f''^{-}(x)$, $K_{xx}^{-}(x,\xi)$, and $\rho_1(x)$. (Received March 12, 1931.)

173. Dr. Wladimir Seidel (National Research Fellow): *On analytic functions with infinitely many zeros.*

Let $f(z)$ be a regular, analytic function defined in the unit circle $|z|<1$. Consider a sequence of points $z_1, z_2, \cdots, z_n, \cdots$ lying in the interior of the unit circle and converging toward the point $z=1$. If $\lim_{n\to\infty}f(z_n)$ exists and is equal to α, we shall call α a *cluster value* of $f(z)$ in $z=1$, and we shall call the set of all cluster values of $f(z)$ in $z=1$ the *cluster set* of $f(z)$ in $z=1$. The author proves the following theorem: *Let $w=f(z)$ be a bounded analytic function: $|f(z)|<1$ in the unit circle $|z|<1$. Let $z_1, z_2, \cdots$ and $z_1', z_2', \cdots$ be two sequences of interior points of the unit circle converging toward $z=1$, such that the non-euclidean distance $D(z_n, z_n')$ is less than a constant M, independent of n: $D(z_n, z_n')<M$, $n=1, 2, \cdots$. Then $f(z)$ always has at least one of the following two properties*: (i) *If $\lim_{n\to\infty}f(z_n)=\alpha$, then $\lim_{n\to\infty}f(z_n')=\alpha$*; (ii) *The cluster set of $w=f(z)$ in $z=1$ contains a region R of the w-plane bounded by a closed Jordan curve and each value w, save at most one, of the region R is assumed infinitely often by $f(z)$ in the interior of $|z|<1$.* (Received March 13, 1931.)

174. Dr. L. S. Kennison: *Linear q-difference equations of continuous order.* Preliminary report.

The equation $\alpha(x)\phi(x)+\int_0^1 L(x, \theta)\phi(\theta x)d\theta=b(x)$ in the unknown function $\phi(x)$ is discussed, and a characteristic equation, in general transcendental, is obtained. Existence theorems for formal series solutions are obtained in this paper in certain cases, most of which apply to the corresponding homogeneous equation ($b(x)=0$). Application is made to Volterra integral equations. (Received March 12, 1931.)

175. Mr. S. B. Myers: *Adjoint systems in the problem of Mayer under general end conditions.*

Necessary conditions in the problems of Lagrange and Mayer with variable end points directly analogous to the Euler equations and transversality conditions in the ordinary calculus of variations problem in the plane have recently been given by Morse and the author (see the Proceedings of the American Academy of Arts and Sciences, vol. 66, No. 6, March 1931). The present paper shows that these necessary conditions can be derived from the criterion for the compatibility of the system adjoint to the differential and terminal equations of variation. The normal case is made a subcase of the abnormal. Such considerations refer a much larger part of the proofs back to the theory of differential equations—in particular, to the theory of adjoint systems—than is usually the case. (Received March 12, 1931.)

176. Mr. J. L. Doob: *The boundary values of an analytic function.*

For brevity, we say that $F(z)$, defined and measurable on $|z|=1$, has the metric cluster value α of the first (second) kind at e^{it} $(0\leqq t\leqq 2\pi)$ if at e^{it} the lower (upper) metric densities of the sets C_ϵ on which $|F(z)-\alpha|<\epsilon$ exceed, for all $\epsilon>0$, a constant $A>0$, independent of ϵ. We say, further, that $f(z)$, supposed analytic for $|z|<1$, has the cluster value α on the radius ρ_t to e^{it}, if $F(z)$ tends to α on some sequence of points on ρ_t tending to e^{it}. The chief results obtained here on cluster values are embodied in the following theorem. Suppose that $f(z)$ is bounded and analytic for $|z|<1$, and that $\lim_{r=1-0}f(re^{it})$, which then exists for almost all t on $(0, 2\pi)$, defines the function $F(e^{it})$. Then (i) if $F(z)$ has a cluster value α of the second kind at e^{it}, $f(z)$ has the cluster value α on ρ_t; (ii) if $F(z)$ has a cluster value α of the first kind at e^{it}, $\lim_{r=1-0}f(r_ne^{it_n})=\alpha$; (iii) if $\lim_{n\to\infty}(r_ne^{it_n})=1$, $(r_n<1)$, and $t_n{}^{p+1}/(1-r_n)$ and $\{1-\text{mes}(I_n\cdot C_\epsilon)/\text{mes}\ I_n\}/(\text{mes}\ I_n)^p$, $p>0$, where I_n is the arc $z=e^{it}$, $0\leqq t\leqq 2t_n$, are uniformly bounded for all n and $\epsilon>0$, then $\lim_{n\to\infty}f(r_ne^{it_n})=\alpha$. It is to be noted that the conditions in (ii) are satisfied almost everywhere on $|z|=1$, and that (ii) contains as a particular case the classical theorem of Pringsheim and others that $F(z)$ cannot have a discontinuity of the first kind (a jump). (Received March 13, 1931.)

177. Dr. Robin Robinson: *On the differential geometry of surfaces in non-euclidean space.*

In this paper an attempt is made to summarize and systematize the classical attack on differential geometry in non-euclidean space, particularly from the

point of view enunciated in W. C. Graustein's article *Invariant methods in classical differential geometry* (this Bulletin, vol. 36 (1930), p. 489). Secondly, a few new theorems on various types of surfaces are proved. Lastly, a discussion of surfaces of Gaussian curvature zero is made from a new point of view. Only for such surfaces does there exist a *characteristic function* whose directional derivatives along the lines of curvature are the geodesic curvatures of those lines. There are found necessary and sufficient conditions that a surface of Gaussian curvature zero exist having a given function as its characteristic function, and these surfaces are classified according to the nature of the functions defining them. (Received March 12, 1931.)

178. Dr. E. J. McShane (National Research Fellow): *On a certain inequality of Steiner.*

Steiner's inequality can be stated thus: If $f_1(x, y), f_2(x, y)$ are functions defined and continuous over the same portion of the xy plane, then the area of the surface $z=\frac{1}{2}(f_1(x, y)+f_2(x, y))$ is at most the arithmetic mean of the areas of $z=f_1$ and $z=f_2$. Conditions are obtained under which the equality is excluded, and applications are developed. The first application is to sequences of surfaces $z=z_i(x, y)$ tending uniformly to $z=z(x, y)$; it is shown that if the area of z_i tends to the area of z, $[(\partial z_i/\partial x-\partial z/\partial x)^2+(\partial z_i/\partial y-\partial z/\partial y)^2]^{1/2}$ converges in measure to zero. The second is to surfaces of least area: if a surface have for a given boundary curve the least possible area, any portion of it which projects orthogonally in a one-to-one way on some plane is analytic, and is a minimal surface in the sense of differential geometry. (Received March 14, 1931.)

179. Professor C. O. Oakley: *Semi-linear equations.* Part I: *Regular polygons.*

An equation of the form $u_0+m_1|u_1|+\cdots+m_n|u_n|=0$, where the m's are constants, and where $u_i=a_ix+b_iy+c_i$, $i=0, 1, \cdots, n$, we define as a semi-linear equation of order $p(\leqq n)$ if there are p and only p non-vanishing coefficients among the m's. The perimeter of any regular polygon, considered as a curve, is given by an equation of this form; and the point set made up of the perimeter of, and all points interior to, any regular polygon is also given by an equation of this same form. Special cases of certain theorems here stated were given without proof by Axel Söderblom (*Sur l'Emploi de Valeurs Absolues dans la Géométrie Analytique*, Göteborg, 1906, 174 pp.). (Received March 14, 1931.)

180. Dr. Jakob Levitzki: *On normal products of algebras.*

If the product AB of two algebras $A=a_1F+\cdots+a_sF$; $B=b_1F+\cdots+b_tF$ over the field F (where A and B are subsets of a given ring P) is a direct right-hand modul over B (that is, if every element of AB can be uniquely expressed as a right-hand linear form with coefficients in B) and if every element a_i is commutative with the algebra B (not necessarily with every element of B) then the product AB is called a normal product with respect to B and is an algebra over F. A necessary and sufficient condition is given that this algebra

be semi-simple. Further, the different representations of a given semi-simple algebra as a normal product of its subalgebras are studied and applications to group-algebras are made. (Received March 12, 1931.)

181. Dr. Oscar Zariski: *On the irregularity of cyclic multiple planes.*

Cyclic multiple planes $z^n=f(x, y)$ are considered, where n is an arbitrary positive integer, and $f(x, y)=0$ is an algebraic curve of order m, possessing nodes and cusps only. A formula for the irregularity $q=p_g=p_a$ of the surface is given in terms of the superabundances of certain linear systems of curves in the (x, y)-plane, determined by simple basis points at the cusps of the curve f. If $f=0$ is an irreducible curve, the following theorem holds: A necessary and sufficient condition that the surface be irregular is that m and n be divisible by 6, and that, putting $m=6j$, the linear system of curves of order $m-3-j$ passing through the cusps of the curve f should be superabundant. If these conditions are satisfied, the irregularity of the surface does not depend on n and is equal to the superabundance of the mentioned system of curves. An application to the theory of cuspidal curves is the following. The cusps of an irreducible plane algebraic curve of order m, possessing nodes and cusps only, impose independent conditions on the curves of any order $m-3-\alpha$ constrained to contain them, as long as α remains less than $m/6$. A noteworthy consequence of this theorem is the existence of sets of 6 non-negative integers which satisfy the Plücker relations and which are not the Plückerian characters of any effectively existent plane algebraic curve. Examples of such illusory Plückerian characters are given here for the first time. (Received March 12, 1931.)

182. Dr. Oscar Zariski: *On illusory Plückerian characters.*

In a paper which will appear in the Annals of Mathematics, the author proves that the well known question as to whether any preassigned set of Plückerian characters always corresponds to an effectively existent plane algebraic curve should be answered in the negative. For instance, there do not exist curves of order 7 with 11 cusps, or curves of order 8 with 16 (or more) cusps. The general theorems on which the proof is based form the subject of another communication (*On the irregularity of cyclic multiple planes*) and involve the elements of the theory of algebraic surfaces. Here a simple direct proof is given of the non-existence of self-dual curves of order 8 with 16 cusps, which makes use only of the elements of plane geometry. (Received March 12, 1931.)

183. Dr. R. P. Agnew (National Research Fellow): *On equivalence of methods of evaluation of sequences.*

We show that if the coefficients in a regular transformation (T): $T_n=\sum_{k=1}^{\infty}a_{nk}s_k$ satisfy certain conditions, then (T) is equivalent to convergence. As an application of these results, we find sufficient conditions for equivalence of (T) and the Cesàro arithmetic-mean transformation. (Received March 25, 1931.)

184. Professor R. Vaidyanathaswami: *The theory of multiplicative arithmetic functions.*

The author presents a theory of the class of functions $f(M_1, M_2, \cdots, M_r)$ of r positive integral arguments, which satisfy the relation $f(M_1N_1, M_2N_2, \cdots, M_rN_r) = f(M_1, M_2, \cdots, M_r)f(N_1, N_2, \cdots, N_r)$ whenever the products $M_1M_2 \cdots M_r$, $N_1N_2 \cdots N_r$ are relatively prime. (Received April 3, 1931.)

185. Professor Jesse Douglas: *The Koebe mapping theorem and the problem of Plateau.*

The mapping theorem of Koebe meant here is that a polyhedral surface can be mapped conformally on a circular disc. The present paper shows that, starting with this theorem, the proof of the existence of a minimal surface bounded by any Jordan contour is a matter of a few supplementary remarks of an evident nature. The paper will appear in the Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität. (Received April 3, 1931.)

186. Professor J. F. Ritt: *Systems of algebraic differential equations.*

This paper describes a theoretical process for resolving a system of algebraic differential equations into irreducible systems. The process consists in differentiating the given system of equations a sufficient number of times and resolving the enlarged system into systems which are irreducible algebraically. A second result of the paper is that, given any system of algebraic differential equations, in unknown functions $y_1, \cdots, y_n$, there exists an equivalent system composed of $n+1$ or fewer equations. (Received March 24, 1931.)

187. Mr. H. L. Dorwart and Professor Oystein Ore: *Certain types of criteria for irreducibility of polynomials.*

Various authors, Schur, Pólya, Stäckel, etc., have devised irreducibility criteria for polynomials, based on the values which these polynomials can take. In this paper is shown how these theorems can be deduced and generalized by means of elementary theorems on the arithmetical properties of polynomials. (Received March 12, 1931.)

188. Professor Warren Weaver: *The Gaussian law of error.*

Most proofs of the Gaussian law of error fall under two types: those which assume that the most probable (or the expected) value following a set of observations is the arithmetic mean of the set; and those which assume the error to be the sum of a very large number of elementary errors. A critical resumé of such proofs is given, pointing out their limitations. Two new proofs of the Gaussian law are then given. The first assumes that, following n observations, the a posteriori probability function for the true value is symmetric with respect to the arithmetic mean of the observations. It is then proved that the error law, if not everywhere discontinuous, is the Gaussian law. The second proof assumes symmetry of the a posteriori probability with respect to some

point x_0 which is an unknown function of the observations. It is again proved that the error law, if not everywhere discontinuous, is Gaussian, and that x_0 is the arithmetic mean of the observations. (Received March 9, 1931.)

189. Professor J. D. Grant: *Second degree addition theorems for even functions.*

It is shown that an even solution of an addition theorem of second degree in $F(x+y)$ is a linear fractional transformation of the Weierstrass "P" function or one of its degenerate forms. (Received March 9, 1931.)

190. Professor H. A. Simmons: *A class of maximum numbers and a class of minimum numbers that are associated with certain Diophantine equations of the form* $\Sigma(1/x_1x_2 \cdots x_r) = b/a$.

For the Diophantine equation (1) $\Sigma(1/x_1x_2 \cdots x_r) = b/a$, $a=(c+1)b-1$, in which c, b are positive integers, the author recently exhibited a "principal" solution $x=w$ (American Mathematical Monthly, March, 1930). If (1) is satisfied by n numbers $x_i=e_i$, $(i=1, \cdots, n)$, where $e_1 \leqq e_2 \leqq \cdots \leqq e_n$, and if the first $n-1$ of these numbers are positive integers, the set e is called a solution ρ of (1). Let $P(x)$ stand for any polynomial that is symmetric in n variables $x_1, x_2, \cdots, x_n$ and has only positive coefficients. Theorem 1. If $x \neq w$ is a solution ρ of equation (1), $P(x) < P(w)$. The author obtains the following generalization of the known result concerning the maximum number that exists in a solution ρ of $\Sigma(1/x_1) = b/a$, a being as in (1): Theorem 2. The w_n of solution w is the largest number that exists in any solution ρ of equation (1) and w_n appears in no solution ρ of (1) except w. In connection with the set $x=W$, where $W_i = [(a/b)\binom{n}{r}]^{1/r}$, $i=1, \cdots, n$, the following is true: Theorem 3. If $x \neq W$ satisfies (1) and has $x_i \geqq 1$, $(i=1, \cdots, n)$, then $P(W) < P(x)$. (Received March 9, 1931.)

191. Professor C. G. Latimer: *On certain divisors of the class number of a cyclotomic field.*

Let F be an algebraic field of degree E and a primary divisor of the field defined by a primitive Mth root of unity. Assume that neither E nor M contains a square factor >1. We shall show that the number of narrowly equivalent ideals in F is divisible by $T = e_1^{k_1-1}e_2^{k_2-1} \cdots e_n^{k_n-1}$, where the e's are the prime divisors of E and each k_i is the number of elements $\neq 0$ in the ith row of a certain matrix (γ_{ij}). (γ_{ij}) is the matrix of the system of linear congruences which define the group to which F belongs. (Weber, *Lehrbuch der Algebra*, vol. 2, p. 100.) It is shown that the classes of ideals in F may be segregated in T genera, each containing the same number of classes. For the case $E=2$, this result is well known. (Received March 10, 1931.)

192. Professor A. E. Young: *Automatic integration and electrical computing instruments.*

Mathematical physicists have developed theories and established equations and formulas which serve as the foundation for various types of measurement of the utmost value in the industrial world. The formulas, almost without excep-

tion, are differential in form, the integrand being expressed as a function of one or more variables whose values, directly measurable, are varying with the time. To use these properly in industrial measurements requires a time determination and summation of the values of the integrand, or in other words, an integration with regard to the time. With a few important exceptions the integration procedure in the past has been replaced by computations based on reference tables and on daily chart records of the values of the time-variables. The author has invented automatic electrical instruments which sum with great frequency and accuracy various types of such integrand elements and thereby, for any particular time-interval, computes the sum whose limit is the time integral for the period. (Received March 10, 1931.)

193. Dr. Gordon Pall (National Research Fellow): *On sums of squares.*

Several results are given concerning sums of three positive squares; for example, every positive integer, except $25\cdot 4^h$, which is a sum of three squares and possesses an odd square factor, is a sum of three positive squares. Application is made of an elegant formula for $r_s(cn)^2/r_s(c)$, where $r_s(t)$ is the number of representations of t as the sum of s squares, and $s=2, 3, 4, 5, 6, 7, 8$, $s=10$ and c not a sum of two squares, $s=11$ and c not a sum of three squares, or $s=12$ and c is even. All numbers not sums of four unequal squares are determined. (Received March 11, 1931.)

194. Dr. Gordon (Pall National Research Fellow): *Improvements of the Cauchy lemma on simultaneous representation.*

The condition $b^2+2b+4>3a$ of the Cauchy lemma on the simultaneous decomposition of an integer a as a sum of four integral squares and an integer b as the sum of the four bases taken $\geqq 0$, is here much improved whether a, b are even or odd. The method is elementary and uses the simplest rational automorphs of a sum of three squares. Some results of the type that certain forms of n possess representations as a sum of three squares with certain special properties are obtained incidentally. (Received March 11, 1931.)

195. Dr. H. P. Thielman: *On the inverse of a certain Volterra functional transformation and its application to the evaluation of some definite integrals involving Bessel functions.*

In this paper formulas are derived which express the integral of products of Bessel functions of any order between certain limits in finite terms of algebraic and trigonometric functions. These results are obtained by the use of a certain Volterra transformation and its inverse, the Laplace transformation. (Received March 11, 1931.)

196. Dr. Wilhelm Maier: *Modular covariants.*

Extending earlier results and methods (*Elementary properties of the t-functions*, Transactions of this Society, vol. 32, (1930)) the author introduces many-

valued analytic functions of two variables, which are closely related to Kronecker's limit formula, and develop their simplest properties. (Received March 11, 1931.)

197. Dr. Wilhelm Maier: *Integral equation of the* Γ*-function.*

Let $\Gamma(s)$ be the classical function of Euler; then there exists a quadratic integral equation, which is satisfied by $\Gamma(s)$. Moreover, that integral equation with initial conditions characterizes the analytic function $\Gamma(s)$ uniquely, and may be taken as the starting point of the Γ-theory. (Received March 11, 1931.)

198. Professor J. W. Alexander and Dr. L. W. Cohen (National Research Fellow): *Homology groups associated with infinite complexes.*

In developing the theory of homology for infinite complexes, there are two definitions of bounding which may be taken as points of departure. These definitions are distinct in the case of cycles with integral coefficients but are equivalent in case the coefficients are residues modulo an integer or are rational numbers. A natural definition of a distance function over the elements of the group suggests itself. The groups are shown to be separable, complete spaces. The modular groups are compact. The groups have denumerable bases. The question of a normal form for the groups is treated and settled in certain cases. (Received March 11, 1931.)

199. Dr. I. Schoenberg (International Research Fellow): *Note on geodesics in higher spaces.*

In recent years geodesics have been studied in the so-called non-holonomic spaces, that is, ordinary Riemann spaces whose curves are constrained to satisfy a given system of Pfaffian equations. There are two different kinds, the mechanical and the geometrical geodesics. The mechanical geodesics are suggested by the dynamical trajectories of mechanical non-holonomic systems and are not the solutions of an ordinary problem of the calculus of variations, while the geometric geodesics have the usual property of such curves in being the extremals of a problem of Lagrange. The author shows in the same way as for geodesics in Riemannian spaces that for these extremals the necessary conditions of Legendre and Weierstrass are always satisfied. To insure a minimum property of the geodesics joining two fixed points it is therefore sufficient to check up Jacobi's condition. This latter condition is also discussed. (Received March 12, 1931.)

200. Mr. A. E. Ross: *On necessary and sufficient conditions for solvability of* $f=0$, *where* f *is any indefinite ternary quadratic form.*

H. J. S. Smith (Proceedings of the Royal Society, vol. 13 (1864), pp. 110–111) stated conditions for solvability in integers not all zero of $f=0$, for a general indefinite ternary quadratic form f, but gave no proof. A. Meyer (Journal für Mathematik, vol. 98 (1885), pp. 177–179) proved Smith's assertion for the

case of forms with the odd determinants. Employing a result of a former paper (this Bulletin, vol. 36, (1930), p. 56), there is given here a proof of the Legendre criterion for solvability of $ax^2+by^2+cz^2=0$ (Legendre, Mémoir, Académie des Sciences de Paris, 1785, pp. 507–514; also see L. E. Dickson, *Introduction to the Theory of Numbers*, p. 117). Using this latter, there is deduced, in a very simple manner, the theorem of Smith without restriction on the determinant of f. (Received March 12, 1931.)

201. Mr. A. T. Craig: *The frequency of x/y for correlated variables.*

Pearson and others have considered the frequency function of x/y and certain approximations have been found for both the correlated and independent cases. In the present paper, the exact frequency laws of x/y, xy, $x+y$, and $x-y$ are obtained when x and y are normally correlated variables. For the special case of x and y independent, the results are valid upon placing $r=0$. When the correlation surface of x and y is of a special non-normal form given by Professor H. L. Rietz in an unpublished paper, the laws of frequency of x/y, xy, $x+y$, and $x-y$ are also given. (Received March 12, 1931.)

202. Mr. Rufus Oldenburger: *Reduced matrices of class* 3.

Certain matrices of class 3 can be associated with general quadratic homogeneous and quadratic Cremona transformations. On the basis of file ranks of such matrices such transformations can be reduced to certain simple forms. The study of such matrices shows that any 2-way couche of a general cubic matrix (a_{ijk}) can under certain conditions be made symmetric under non-singular transformations and reduced to canonical form depending on the rank of this couche. The conditions under which a further couche can be reduced to canonical form without changing the couche already reduced are treated for cases in which (a_{ijk}) is of range 3 and 4, corresponding to plane and space transformations, and for a general range n where this couche is of any file rank. The geometric properties of certain reduced transformations are studied. It is shown that if $n=3$ a couche of rank one determined by fixing one of the symmetric indices can never be a matrix associated with a Cremona transformation. (Received March 12, 1931.)

203. Professor H. W. March: *The elastic instability of members with thin outstanding flanges of non-isotropic material.*

The methods used by Timoshenko in discussing problems of elastic instability for structural members of isotropic material are extended to members of non-isotropic material such as wood having three mutually perpendicular planes of elastic symmetry. A flange is treated as a rectangular elastic plate having the two opposite edges to which the compressive load is applied simply supported, one edge free, and the remaining edge either perfectly fixed, partially fixed or simply supported. This mathematical investigation forms part of a report on elastic instability prepared in collaboration with G. W. Trayer at the U. S. Forest Products Laboratory for the National Advisory Committee for Aeronautics. (Received March 12, 1931.)

204. Dr. Leonard Bristow: *Expansion theory associated with linear differential equations and their regular singular points.*

This differential equation is taken in the form $x^{n-1}y^{(n)}+x^{n-2}ay^{(n-1)}+x^{n-3}\rho_2(x)y^{(n-2)}+\cdots+\rho_n(x)y/x+\lambda y/x=0$, where $\rho_2(x), \cdots, \rho_n(x)$ are analytic functions of the complex variable x at $x=0$. The problem is set up as a boundary value problem. Suitable boundary conditions are determined together with characteristic values of λ. An expansion of $(t-x)^{-1}$ is obtained in a series of the solutions of the differential equation with argument x. The coefficients are polynomials in t^{-1} which are associated with a corresponding solution of the adjoint equation. By means of Cauchy's theorem there are obtained the expansions of (a) an arbitrary function analytic at the origin, (b) an arbitrary function analytic in an annular ring, (c) an arbitrary function of more than one complex variable analytic in suitable regions. The expansions converge absolutely and uniformly with respect to x in regions bounded by circles. (Received March 13, 1931.)

205. Professor H. S. Wall: *General theorems on the convergence of sequences of Padé approximants.*

Theorem I. Let $P(z)$ be a convergent series of Stieltjes. Then every infinite sequence of distinct approximants converges to $P(z)$ uniformly over K, where K is an arbitrary closed region lying entirely within the circles of convergence of $P(z)$ and of its reciprocal. Theorem 2. If $P(z)$ is a series of Stieltjes which is summable (Borel) to $F(z)$, then the diagonal files S_k, $k\geqq -1$, converge to $F(z)$. The same holds for $k<-1$ if the reciprocal of $P(z)$ is summable. Theorem 3. If $P(z)$ is a convergent positive definite series, then $S_k(k=0, \pm 1, \pm 2, \cdots)$ converges to a common limit $F(z)$. It is shown that the Stieltjes summability (as applied to positive definite series) is more inclusive than Borel summability. (Received March 14, 1931.)

206. Professor W. S. Kimball and Mr. W. J. King: *Theory of heat conduction and convection from a hot vertical plate.*

An analysis is made of free conduction and convection in air opposite a hot vertical plate, and checked by recent experiments of German investigators. Two new approximate empirical laws are employed to supplement the slightly modified classical theory. Law I: The locus of the maxima of the convection velocity curves is an isothermal surface: $x=(T_1-T_0)/[2a(1+be^{-ay})]$, at temperature half way between T_1, the hot plate temperature, and T_0, the ambient air temperature. Law II: Half the heat is convected up away inside this surface, or "film" boundary, and half outside. This theory is consistent with the five-fourths power law for variation of heat transfer, and with known facts concerning maximum convection velocities, heat transfer coefficients, and variations of film thickness. Two theorems concerning rate of heat transfer by convection and viscosity effects are given. (Received March 24, 1931.)

207. Professor H. K. Hughes: *On the analytical extension of functions defined by factorial series.*

This paper is concerned with the analytical extensions of functions such as are defined by series of the forms (1) $\sum_{n=0}^{\infty} g(n)/[z(z+1)(z+2)\cdots(z+n)]$ and

(2) $g(0)+\sum_{n=0}^{\infty}g(n)(z-1)\cdot(z-2)\cdots(z-n)$. It is well known that the region of convergence of a series such as (1) or (2) is in general a half-plane, lying to the right of a vertical line in the z complex plane. The author obtains expressions which, under certain restrictions on the general coefficient $g(n)$, furnish the analytical extension of the function defined by the given series (1) or (2) throughout the whole finite z-plane, the neighborhoods of the points $z=0$, -1, -2, $\cdots$ being excepted. (Received April 4, 1931.)

208. Reverend J. E. Case: *The behavior of the Hessian of a curve at a singular multiple point of the curve.*

So far as known, there is no literature on the general case as treated in this paper. Certainly the method of attack is new. The object of this paper is to find a method of placing the Newton polygon of the Hessian of an algebraic plane curve defined by $f(x, y)=0$ on the analytic triangle and by comparing it with the Newton polygon of the curve to discover the relationship between the curve and its Hessian at a multiple point of the curve. The method is based on a summation form of writing the Hessian; this is thought to be original. Within the limits of this summation all the points of the analytic triangle which contribute to the Newton polygon will be found. A knowledge of the terms of $f(x, y)$ which contribute to its Newton polygon is sufficient to determine the Newton polygon of the Hessian. There is a definite relationship between the branches of the Hessian and the branches of the original curve. The nature of the Hessian at a multiple point of its original curve is completely known when the degrees of the terms which contribute points to the Newton polygon of the curve are known. (Received April 4, 1931.)

209. Mr. Selby Robinson: *Note on spaces satisfying the first enumerability axiom of Hausdorff.*

In this paper it is proved that the first enumerability axiom and the first three properties of Riesz together constitute a necessary and sufficient condition that a space be a space V as defined by Fréchet in *Espaces Abstraits*. In a space V the following properties are shown to be equivalent: the fourth property of Riesz, property D of Hausdorff, and the property of being an L space. It is shown by an example that a space V need not be an L space. If a point has a well ordered decreasing family of neighborhoods equivalent to the family of all neighborhoods of the points, sets of conditions are given which are sufficient to insure that this decreasing family can be replaced by an enumerable sub-family. An example is given of a separable, compact, completely normal space which satisfies the first enumerability axiom but not the second. (Received April 4, 1931.)

210. Miss I. M. Schottenfels: *The holoedric isomorphism of the collineation group G_{20160}, $P.G[2, 2^2]$, and the ternary linear fractional group G_{20160} in $G.F.[2^2]$.*

U. G. Mitchell in his doctor's dissertation (Princeton, 1910) designates the group of all projective transformations in $P.G[2, 2^2]$ as G_{60480}; its self-conjugate sub-group of determinant unity is the simple group G_{20160}. The author proves

that this group is holoedrically isomorphic with the ternary linear fractional group in the Galois field $[2^2]$. (Received April 4, 1931.)

211. Dr. C. C. Craig (National Research Fellow): *On the composition of dependent elementary errors.*

If the probability variable x is the sum of n equal and independent component probability variables, $z_1, z_2, \cdots, z_n$, Cramér has shown that under certain quite general conditions the representations of the probability and frequency functions of x by a Charlier type A series (series in Hermite polynomials) actually has asymptotic properties with respect to n. (Skandinavisk Aktuarietidskrift, 1928, pp. 13–74). It is well known that under very wide conditions the probability function of x approaches the Gauss-Laplace law as n increases indefinitely. Bernstein has shown that this latter fact still holds when the condition of independence on $z_1, z_2, \cdots, z_n$ has to a considerable extent been removed (Mathematische Annalen, 1927, pp. 1–59). The present paper, by a combination of the methods of these two investigators, shows that the type A representation of the probability and frequency functions of x still retains its asymptotic property when the independence condition on $z_1, z_2, \cdots, z_n$ has to a certain extent been removed. (Received March 17, 1931.)

212. Dr. C. C. Craig (National Research Fellow): *Note on the distribution of means of samples of N drawn from a type A population.*

Recently Dr. George A. Baker has found the distribution of means of samples drawn at random from a population whose frequency function is a Gram-Charlier series of a finite number of terms. (Annals of Mathematical Statistics, vol. 1, pp. 199–204). It is the purpose of this note to call attention to the fact that by the use of the semi-invariant notation, Dr. Baker's results may be reached much more simply and directly. (Received March 17, 1931.)

213. Dr. C. C. Craig (National Research Fellow): *Sampling in the case of correlated observations.*

There are many important cases in which the usual assumption of independence or randomness in the set of observations which compose a sample of N is not justifiable. In the present paper, by the use of the semi-invariants of Thiele, a method is developed for the calculation of the characteristics of the distribution of moments about the mean in such samples, which is particularly adapted to the case in which the N observations are normally correlated. General formulas for computation in this case are given. These are applied to the particular case in which only consecutive observations are correlated and in a constant degree and results are given in addition to the mean and variance of the variance given by Rhodes (Journal of the Royal Statistical Society, 1927, pp. 135–143). A by product is a proof that if all the observations are normally correlated with each other and in equal degree, the distribution of all the moments about the mean in samples of N is the same in form as if the observations were independent. (Received March 17, 1931.)

214. Dr. C. C. Craig (National Research Fellow): *On a property of the semi-invariants of Thiele.*

Given a linear form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

of a set of probability variables, $x_1, x_2, \cdots, x_n$, it is one of the most useful and important properties of the semi-invariants of Thiele that the rth semi-invariant of this form is, in the case where $x_1, x_2, \cdots, x_n$ are independent, simply

$$a_1{}^r\lambda_r{}^{(1)} + a_2{}^r\lambda_r{}^{(2)} + \cdots + a_n{}^r\lambda_r{}^{(1)},$$

in which $\lambda_r{}^{(i)}$ is the rth semi-invariant of x_i. This paper shows that among such isobaric functions of the moments of several variables this property is characteristic of the semi-invariants. Further, it is also shown that the necessary and sufficient condition that there be other such functions of weight $\geqq k$ in special cases is that the probability functions of each x_i, in case of independence, be such that, for each, λ_r vanishes for some $r < k$. (Received March 17, 1931.)

215. Professor Raymond Garver: *Determinants and the real roots of an equation.*

Borchardt and Weber have shown that the number of distinct roots, and of distinct real roots, of an equation with real coefficients can be stated in terms of the rank and the signs of the principal minors of certain symmetric determinants. Borchardt's determinant is the usual determinant form of the discriminant of the equation, while that used by Weber has lower order by one. In this paper, another such determinant is developed, with the aid of which the relation between the other two, which is not at all obvious, becomes apparent. (Received March 23, 1931.)

216. Professor Harold Hotelling: *The generalization of Student's ratio.*

The problem of simultaneous estimation of deviations in several variates has been attacked by Pearson and by Romanovsky with several "coefficients of racial likeness," but these are valid only on the assumption that the variates are independent, and they have excessively complicated sampling distributions. The need for simultaneous estimates extends outside of biometry, and includes the problem of comparing correlated variates such as prices at different times of a list of articles. The author recommends a new measure T of simultaneous deviations which, for a single variate, reduces to Student's ratio of mean to standard error. Putting a^{ij} for the covariance *estimated from the sample* of the statistics ξ^i and ξ^j (means, regression coefficients, etc.), a_{ij} for the tensor conjugate to a^{ij}, and n for the number of degrees of freedom among the columns of the reduced matrix of observations, we define the proposed statistic by

$$nT^2 = \sum\sum a_{ij}\xi^i\xi_0.$$

Its invariantive character shows that its distribution is independent of the covariances in the population, and this facilitates the derivation of this distribution, which is shown to be reducible to the incomplete beta function and hence to Fisher's distribution for the analysis of variance. (Received March 23, 1931.)

217. Dr. D. H. Lehmer (National Research Fellow) and Mr. R. E. Powers: *On factoring large numbers.*

Various non-tentative methods of factoring the odd number N based on the expansion of $N^{1/2}$ in a regular continued fraction have been described. Most of the methods that have been suggested depend for their success upon the appearance of a perfect square among the denominators of the complete quotients of even order. Two methods are now available when the product of any number of these denominators is a square, an event that occurs quite frequently in practice. The object of each method is to obtain a difference of squares divisible by N. To do this the first method uses the numerators of the complete quotients, and the second method the numerators of the convergents. These methods are described, compared and illustrated. Proof is also given of the fact that the success of one method implies the success of the other, so that one is free to use whichever method is easiest to apply. (Received March 18, 1931.)

218. Professor A. D. Michal: *Concerning certain one-parameter continuous functional groups and their integro-differential invariants.*

The first part of the paper is concerned with the functional invariants $f[y'(x), y(a), z]$ under the group composed of the one-parameter Volterra group in $y(x)$ and the translation group $z_1 = z + \tau$. Extensions to the case of integro-differential invariants of order n are also taken up. The second part of the paper deals with functional invariants of one-parameter functional groups arising from Cauchy problems of certain second-order partial differential equations of mathematical physics. (Received March 23, 1931.)

219. Professor A. D. Michal: *A theory of integral invariants in composite functional spaces.*

This paper is concerned with a theory of integral invariants in the composite functional spaces whose points have coordinates $(y_1^\alpha, y_2^\alpha, \cdots, y_n^\alpha, t)$. The paper will appear in an early issue of the Proceedings of the National Academy of Sciences. (Received March 23, 1931.)

220. Mr. Saul Pollock: *On the determination of line loci in four-space by a graphic method in geometry.*

In Wong's paper *On certain loci of lines incident with curves and surfaces in four-space* (this Bulletin, vol. 35 (1929), pp. 353–358) a list of 27 formulas for the orders of the ruled loci whose lines satisfy four, five and six simple conditions is derived by a synthetic method. It is the purpose of this paper to show how a new method (graphic method) may be employed to derive the formulas given by Wong and formulas for the special cases (where they exist) for each of the 27 formulas mentioned and additional formulas not included in the list. The graphic method is a method by decomposition applicable to general varieties in S_n. It was first introduced by Wong in a paper *On the number of apparent triple points of surfaces in space of four dimensions* (this Bulletin, vol. 35 (1929), pp. 339–343). (Received April 6, 1931.)

221. Professor A. A. Shaw: *Solutions of homogeneous linear difference equations by means of infinite determinants.*

It is the purpose of this paper to give solutions of homogeneous linear difference equations by use of infinite determinants. From a given homogeneous linear difference equation we derive an infinite system of equations with infinitely many unknowns. We solve this system by infinite determinants, using the "heuristic" method. The first part of the paper gives the solution of $a_x u_{x+2}+u_{x+1}+a_x u_x=0$, where a_x is any arbitrary function of x. The laws of expansions of the determinants are proved by mathematical induction and their convergence by use of von Koch's lemma. The second part of the paper deals with the solution of $b_x u_{x+2}+u_{x+1}+a_x u_x=0$, where a_x and b_x are arbitrary functions of x, treated in the same manner as the one in part I with the solution of an example to illustrate the general theory. Part III considers the solution of part of the problem in part II, as $n\to-\infty$ instead of $+\infty$. The last part of the paper gives the solution of $c_x u_{x+3}+b_x u_{x+2}+u_{x+1}+a_x u_x=0$. The method is quite general and is applied to equations of order n. (Received March 14, 1931.)

222. Professor J. V. Uspensky: *A problem in the geometry of numbers.*

A manifold Γ of $n-s$ dimensions is determined in a space of n dimensions by a set of s linear (independent) equations $\sum_{j=1}^{n} c_{ij}x_j=d_i$, $(i=1, 2, 3, \cdots, s)$. If d_i runs independently over all integral values the manifolds Γ form a discrete set. The following question was raised by Professor Harald Bohr in connection with a certain problem in the theory of Diophantine approximations: What upper limit can be set for a minimum distance of a given point from the system of manifolds Γ? In the present paper an answer is given to this question. (Received March 23, 1931.)

223. Mr. W. M. Rust, Jr.: *Integral equations and the cooling problem.*

The first part of this paper deals with a method of solving the generalised Abel integral equation $\int_0^t [G(t, t')u(t')/(t-t')^{\lambda}]dt'=f(t')$, $0<\lambda<1$, for a certain type of absolutely continuous function $G(t, t')$. The function $f(t)$ is *not* assumed to vanish at $t=0$. A Volterra integral equation of the second kind is obtained that has an absolutely continuous solution whose derivative satisfies the preceding equation nearly everywhere.

The remainder of the paper applies this method to the problem of the linear flow of heat in a body composed of several materials. The boundary conditions are the usual ones, as given by Riemann-Weber, *Differentialgleichungen der Physik*, 1912, vol. 2, p. 85. A uniqueness theorem is obtained for general forms of solution, and such solutions are obtained in terms of rapidly converging series of integrals. (Received April 5, 1931.)

224. Dr. R. P. Agnew: *On complex methods of summability.*

This paper raises and discusses the following and related questions: If a method of summability evaluates a complex sequence $\{u_n+iv_n\}$ to $u+iv$, does that method evaluate $\{u_n\}$ to u? (Received April 13, 1931.)

225. Dr. A. B. Brown: *Topological invariance of sub-complexes of singularities.*

We consider the question of topologically invariant sub-sets of a given complex. In the first part of the paper we consider sub-complexes which are determined from the incidence relations by simply counting, and here obtain results of which the following is a simple example: If a complex K is an n-cycle (mod 2), then so is every complex homeomorphic to K. In the second part of the paper, we consider sub-sets (which are sub-complexes) defined in simple terms, but such that it is not known at present how to determine them in all cases from the incidence relations. Given two homeomorphic complexes K and K', in terms of these sub-sets we state a necessary and sufficient condition that the image of a given point of K be on a cell of K' of dimension not exceeding i, under all possible homeomorphisms between K and K'. The proofs depend on the Brouwer theorem of invariance of regionality. (Received April 13, 1931.)

226. Professor V. G. Grove: *The transformation C of nets in hyperspace.*

In this paper the author extends the notion of the transformation C of nets in ordinary space to nets in spaces of dimensions greater than three. An extension is made of the notion of lines in Green's relation R with respect to nets in ordinary space to nets in hyperspace. As in the three dimensional case there is a close relation between the transformation C of nets and the relation R between lines with respect to the net. The derived lines of the congruence G associated with the transformation C intersect the tangents to the curves of the nets in certain covariant points found by Bompiani. Conversely, having given a net possessing these covariant points, all C transforms of the net may be determined by making use of the relation R. (Received April 13, 1931.)

227. Dr. R. S. Underwood: *On universal quadratic null forms in five variables.*

By a transformation due to L. E. Dickson we may reduce the general quadratic null form to the form $F=2^e gaxy+gby^2+cyz+gd\psi(z, w, \cdots)$ where g and a are odd, a is prime to d, c is prime to g, and the greatest common divisor of the coefficients of ψ is 1. In the case of five variables, $\psi=\alpha(hz^2+jzw+lw^2)+Azv+Bwv+Cv^2$. We then apply the following lemma: If each of the congruences: (1) $F\equiv G \pmod{gay}$, and (2), $F\equiv G \pmod{2^e}$, where G is an arbitrary integer, has a solution such that y is odd, then $F=G$ is solvable. This paper derives the conditions necessary and sufficient for the solvability of (1) in the five variable case, and also discusses the briefer results for (2). Auxiliary results are proved which allow greater freedom in the choice of y than was required for the complete solution of the three and four variable cases. (Received April 25, 1931.)

228. Mr. L. B. Robinson: *On a lacuna in the theory of systems of partial differential equations. Part II.*

In a previous notice in this Bulletin, the author considered the system (A): $\partial u/\partial x_i=f_i(x_1, x_2, x_3; u)$, $(i=1, 2, 3)$, and investigated relations connecting the

independent variables when a common solution exists. He assumed that $f_2 \neq 0$, $f_3 \neq 0$ in the neighborhood of the initial conditions. This restriction is needless. Consider two equations (B): $\partial u/\partial x_i = f_i(x_1, x_2; u)$, $(i=1, 2)$. The initial conditions are $u = x_i = 0$, $f_i(0, 0, 0) = 0$. A common solution exists when a relation to be calculated exists between the x_i. $x_2 = \eta(x_1)$, $x_1 = \eta_{-1}(x_2)$. Calculate the solution of the first equation of (B) by successive approximations. Solve the second equation by the same method. To assure equality of the two solutions, solve an integro-differential equation by a convergent series. To demonstrate convergence, consider (1): $\mu x Y + \lambda \int Y dx = x^2 P(xY) + Y^2 Q(xY)$. If certain inequalities are satisfied, the above has a converging solution. It is easy to construct a series like (1) available as a dominant series to prove that the integro-differential equation in question can be solved by a converging series. Extension of the above results to three or more equations is easy. (Received April 27, 1931.)

229. Dr. Gordon Pall (National Research Fellow): *On representation by positive binary quadratic forms.*

In this paper a method is developed of finding a simple closed formula for the number of representations of n in *any* positive integral binary quadratic form *whatever*. Let $f_1, \cdots, f_n$ denote a representative system, consisting of one form from each class, of positive, primitive, integral, binary, quadratic forms of a given discriminant $d < -4$. Let $f_i(n)$ denote the number of representations in f_i of n. The function $r(n) = \frac{1}{2}\{f_1(n) + \cdots + f_h(n)\}$, $(n = 0, \pm 1, \pm 2, \cdots)$, is factorable and well-known, if n is prime to d, and its value being tabulated in the present paper, incidentally, for general n. While not generally factorable, the individual functions $f_i(n)$ satisfy reduction formulae for $f_i(p^a m)$, which are discovered and established here, the type of formula depending only on which of the forms f_j represents p. Combining these formulae gives noteworthy expressions for the number of representations in each f_i. Many beautiful relationships are found among the $f_i(n)$. These results give a concrete expression to the theory of composition. A fuller abstract of these results will appear very soon in the Proceedings of the National Academy of Sciences. (Received May 4, 1931.)

230. Dr. Emma Whiton McDonald: *Magic cubes which are uniform step cubes.*

D. N. Lehmer has applied the theory of congruences to the study of magic squares. In the present paper this theory has been extended to the subject of magic cubes which are uniform step cubes. The magic squares contained in any cube are examined and criteria developed for the determination of the nature of these squares. (Received May 6, 1931.)

231. Professor A. F. Carpenter: *A triad of ruled surfaces defined by reciprocal polars.*

The flecnode curve C of a ruled surface R cuts each line element l of R in two points. The osculating planes of C at pairs of points so determined intersect in lines whose totality constitutes a second ruled surface. For that line of this surface which corresponds to l there is determined its polar reciprocal with respect to the linear complex osculating R along l. The totality of these lines

constitutes a third ruled surface. The paper is concerned with an investigation of the properties of this triad of surfaces. (Received May 8, 1931.)

232. Dr. D. H. Lehmer (National Research Fellow): *A ternary analogue of an abelian group.*

We study in this paper the fundamental properties of a class K of abstract marks subjected to a ternary operation which may be written $a \cdot b \cdot c$. If two postulates (similar to those of Hurwitz for an Abelian group) hold, the class K is called a triplex under the operation. If it is possible to interpret $a \cdot b$ as an element of K then the triplex becomes an ordinary Abelian group under $\cdot$, and is called the extension of the group. The extension of every Abelian group is a triplex, but some triplexes are not the extension of any group. A triplex in general has no unit element. Instead there is associated with each element a, an element a' for which $a \cdot a' \cdot b = b$ for every b. If there is an element e for which $e' = e$, then the triplex is an extension of an Abelian group whose unit is e and conversely. In spite of this lack of a unit, the theory of triplexes of finite order parallels closely that of abelian groups. Precise analogues exist for the theorems of Lagrange and Cayley on subgroups but not for Sylow's theorem. Although every triplex has a set of generators it need not have a basis, or a subtriplex. Simple examples of triplexes such as the set of all odd integers under addition of the set $(i, -1)$ under multiplication, are readily constructed. (Received May 9, 1931.)

233. Professor Dunham Jackson: *Note on the application of Markoff's theorem to problems of approximation in the complex domain.*

In a recent note (this Bulletin, vol. 36 (1930, p. 851) the author has discussed an extension to the complex domain of a method previously used in connection with problems of the approximate representation of real functions, in which the proof of convergence is based on Bernstein's theorem on the derivative of a polynomial or trigonometric sum. The discussion for the case of a complex variable involved certain restrictions on the boundary of the region with which the problem was concerned. The object of the present paper is to show how these restrictions may be somewhat lightened, at the expense, to be sure, of a compensating increase in the stringency of the hypotheses on the function to be approximated, by the use of Markoff's theorem on the derivative of a polynomial in place of that of Bernstein. (Received April 30, 1931.)

234. Professor A. A. Albert: *On direct products.*

The principal contribution of this paper is a set of theorems on the direct product of a normal division algebra and an algebraic field over the same reference field, with applications of the Galois theory of equations. In particular it is shown that it is possible to extend the reference field of a normal division algebra of order p^2, p a prime, such that the algebra over the extended field is a cyclic (Dickson) normal division algebra over this new field. These results are applied to give an independent new proof of a little known theorem of R. Brauer which reduces the problem of determining all normal division algebras of order n^2 to the case where n is a power of a prime. (Received May 6, 1931.)