ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THIS SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume.* Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

1. Professor Clifford Bell: *On triangles in- and circumscribed to two cubic curves.*

This paper deals with two cubic curves so related that one curve passes through the vertices of a given triangle while the other is tangent to each of the sides of this same triangle. A study is made of the set of equations whose solutions give the triangles in- and circumscribed to the two given curves. Special cases are considered when the cubic degenerates. (Received November 8, 1930.)

2. Professor E. T. Bell: *All recurring series having multiplicative periodicity.*

In a note in the Proceedings of the National Academy of Sciences (November, 1930), the author determined all recurring series having additive periodicity. In this note, to appear later in the Proceedings, the like is done, by a precisely similar argument, for multiplicative periodicity. (Received November 6, 1930.)

3. Professor E. T. Bell: *Modular interpolation.*

In the theoryof power residues, also somewhere in arithmetic, it is frequently necessary to construct $f(x)$ such that $f(x) \equiv r_i$ mod m when $x \equiv a_i$ mod m, $(i = 1, \cdots, t)$, the m, r_i, a_i being given integers. Simple forms of $f(x)$ are given in this problem and its obvious generalizations. (Received November 6, 1930.)

4. Professor E. T. Bell: *Singular relations between certain arithmetical functions.*

Let $a_1, \cdots, a_r$ be numerical constants not all zero, and let $f_1(t), \cdots, f_r(t)$ be r distinct arithmetical functions of t. Then, if and only if $a_1f_1(t) + \cdots + a_rf_r(t) = 0$ has only a finite number of integer solutions $t > 0$, the relation $a_1f_1(t) + \cdots + a_rf_r(t) = 0$ is here defined to be singular for $f_1(t), \cdots, f_r(t)$. A complete set of singular relations is determined for the functions giving the

* See pp. 1–2 and p. 45 of the January, 1930, issue.

numbers of representations of an integer in each of the forms $x^2+y^2+z^2+w^2$, x^2+y^2, $x+2y^2$, $x^2+y^2+z^2$, x^2+3y^2, $x^2+y^2+2z^2$, and all integers satisfying the relations are stated. (Received November 6, 1930.)

5. Professor E. T. Bell: *Factorability of numerical functions.*

If f, g, h are numerical functions such that $\Sigma f(d)g(\delta)=h(n)$, $(n=d\delta)$, for all integers $n>0$, we write $fg=h$, which defines the D-product fg of f, g. If for all coprime m, n, $f(mn)=f(m)f(n)$, f is called factorable. The following is proved: If $fg=h$, and h is factorable, then both of f, g are factorable or neither is. The converse is false, but if both of f, g are factorable so is h. If $f(1)\neq 0$, f is called regular. In an obvious manner $f^a g^b \cdots h^c$, where $a, b, \cdots, c$ are rational numbers and $f, g, \cdots, h$ are numerical functions, is defined as for the special case fg. If $f, g, \cdots, h, k$ are any numerical functions, of which $f, g, \cdots, h$ are regular and k is factorable, such that $f^a g^b \cdots h^c=k$, then all or none of $f, g, \cdots, h$ are factorable. (Received November 6, 1930.)

6. Professor E. T. Bell: *On a type of illusory theorem concerning higher indeterminate equations.*

In the literature of higher arithmetical forms, Liouville, Gegenbauer and others have stated theorems which, when examined, are illusory in the following sense: the theorems direct us to perform unnecessary and prolix tentative calculations to solve the problems to which the theorems refer, and presuppose the complete solutions of the problems. As examples may be cited the theorem of Liouville, Journal de Mathématiques, (2), vol. 4(1859), pp. 271–2, stated without proof; the proof reveals the illusory character of the theorem; Gegenbauer's theorems, Wiener Sitzungsberichte, vol. 95, II (1887), pp. 606–9. (Received November 6, 1930.)

7. Professor E. T. Bell: *Note on functions of rth divisors.*

The rth divisors defined by D. H. Lehmer (American Journal of Mathematics, vol. 52 (1930), pp. 293–304) are here considered in relation to the theory of D-multiplication (see, for example, *Algebraic Arithmetic*, Colloquium Publications of this Society, vol. 7, 1927). The inversions and other processes for rth divisors are immediate consequences of D-multiplication by the function u, where $u(n)=1$ for all integers $n>0$. (Received November 6, 1930.)

8. Professor E. T. Bell: *Functional equations of totients.*

The Jordan totient $\phi_r(n)$ of order r (r any real or complex number) is uniquely defined by its functional equation $\Sigma\phi_r(d)=n^r$, where Σ refers to all divisors $d>0$ of the arbitrary integer $n>0$. This is a functional equation in algebra D (see *Algebraic Arithmetic*, 1927). Two remarkable new functional equations whose unique solution is $\phi_r(n)$ are derived in algebra L, L being the L.C.M. process of multiplication and summation recently devised by D. H. Lehmer, an account of which will appear in the American Journal of Mathematics. (Received November 6, 1930.)

9. Dr. Leonard Carlitz (National Research Fellow): *Arrays of numbers.*

Arrays of numbers (generalizing the Pascal and Stirling triangles) are here considered. First certain special arrays are introduced. Explicit expressions for the elements of a large class are then set up. The properties of the arrays are easily deduced from associated difference operators. (Received November 6, 1930.)

10. Dr. Leonard Carlitz (National Research Fellow): *On the discriminantal divisors of Abelian fields.*

In this paper Abelian fields are classified in a simple manner; their discriminants are then computed and compared. In addition certain results on index divisors are obtained. (Received October 6, 1930.)

11. Dr. Leonard Carlitz (National Research Fellow): *A modular form associated with a cubic field.*

By examining the functional equation of the zeta function and similar functions of an algebraic field, Hecke (Mathematische Annalen, vol. 97 (1926), pp. 210–242) has indicated a method of constructing modular forms associated with the rational Dirichlet L-functions and the Dedekind zeta-function of both real and imaginary fields. In this paper is indicated the construction of a modular form associated with an arbitrary cubic field of negative discriminant; it is derived from the quotient of the zeta-function of the field by the Riemann zeta. (Received November 5, 1930.)

12. Professor P. H. Daus: *A condensed table of linear forms.*

Tables of linear forms are lists of arithmetical progressions with modulus $2D$ or $4D$, such that if e is any prime number in the table, then D is a quadratic residue of e. The number of entries is $\frac{1}{2}\phi(D)$, $2\phi(D)$, or $\phi(D)$, according as $D\equiv 1$, 2, or 3 modulo 4. If we use the properties that if e_1 and e_2 are numbers of the linear form, then e_1e_2 and e_1^k are also, and a table of odd primitive roots of primes, a minimum number of entries can be listed from which all entries may be obtained. If $D=\pm$a prime, the required single entry is obtained directly. If D is composite, the determination of a minimum set of entries depends upon two facts. If D is the product of two primes, p and q, and (1) if $p-1$ and $q-1$ have no common factor other than 2, a common primitive root may be found by solving the linear diophantine equation $r+2p=s+2q$, where r and s are odd primitive roots of p and q; (2) if $q-1$ is a factor of $p-1$, then the set r, $r+2p, \cdots, r+(q-1)p$ yields the required minimum number of entries. (Received November 4, 1930.)

13. Professor Glenn James: *On the limit of an arc of a one-parameter curve.*

Assuming that on a given interval the curve $y=f(x, r)$, $r<R$, and its limit curve are of bounded total variation, possess first derivatives almost every-

where and the latter possesses second derivatives, always zero or with only a finite number of zeros, this paper establishes a necessary and sufficient condition for the non-equivalence of the limit of the arc of $y=f(x, r)$, in its given interval, and the corresponding arc of the limit curve. Upon the basis of this condition a method of constructing explicit examples is formulated. (Received November 5, 1930.)

14. Professor A. D. Michal: *Numerical functional tensors.*

This paper is concerned with the analogs of the Kronecker delta and generalized Kronecker deltas in function space. (Received November 8, 1930.)

15. Professor A. D. Michal: *Partial difference equations with variable coefficients arising in the theory of functional forms.*

In this paper the theorem is proved that the number $N(p, m)$ of distinct function coefficients in a general m-parameter family of multilinear functional forms (continuous of order zero) satisfies the difference equation

$$N(p,m) = N(p-1,\ m+1) + mN(p-1,\ m).$$

The author is greatly indebted to his colleague Professor E. T. Bell for deriving the general solution of this equation. More complicated partial difference equations and systems of partial difference equations arise in the theory of functional forms in a composite range and in the theory of functional forms of continuity order r. These equations are set up and solved. (Received November 8, 1930.)

16. Professor A. D. Michal: *Concerning linear integro-differential forms.*

This paper develops an invariant theory of linear integro-differential forms. Infinite groups of functional transformations involving partial derivatives of kernels of linear functional groups play a central role in this theory. (Received November 8, 1930.)

17. Professor A. D. Michal: *Projective integral invariants attached to the trajectories of differential systems.*

The differential system (1) $dx^\alpha/d\tau = \xi^\alpha(x^1, x^2, \cdots, x^n)$, $(\alpha=0, 1, 2, \cdots, n)$, with $\xi^0\equiv 1$ is invariant under a certain group G of coordinate transformations in $x^0, x^1, \cdots, x^n$. This paper is concerned with projective integral invariants of the system (1) in the space of $(x^0, x^1, x^2, \cdots, x^n)$. The theorem is proved that a necessary and sufficient condition that I be a projective integral invariant (absolute or relative) attached to the trajectories of the system (1) is that I be a Cartan complete integral invariant. The conditions for integral invariance are written in the form of projective tensor equations. (Received November 8, 1930.)

18. Dr. Gordon Pall (National Research Fellow): *On sums of four or more values of $\mu x^2+\nu x$ for integers x.*

Let $0<\nu<\mu, f(x)=\mu x^2+\nu x$. Let all sums of four values of $f(x)$ for integers x be arranged in order of magnitude. Then the largest gap between consecutive entries in the table so obtained is $\mu-\nu$ if $\mu\geqq\frac{3}{2}\nu$; $5\nu-3\mu$ if $\mu\leqq\frac{3}{2}\nu$. (Received October 31, 1930.)

19. Dr. Gordon Pall (National Research Fellow): *On sums of two or four values of a quadratic function of x for integers x.*

This paper gives a fairly comprehensive determination of the largest gap in the table of all sums of four values of a quadratic function of x with real coefficients for integers $x\geqq w$, w given. For sums of two values the problem is trivial since there is no largest gap. (Received October 31, 1930.)

20. Professor A. A. Shaw: *H. Von Koch's first lemma and its extensions.*

In this paper the author proves independently von Koch's first lemma which appeared in Comptes Rendus, vol. 120 (1895), p. 144, without proof in the following form: $\alpha_1, \alpha_2, \cdots$ and $\beta_1, \beta_2, \cdots$ being any given quantities whatever, the necessary and sufficient condition for the absolute convergence of the infinite determinant

$$(1)\qquad \begin{vmatrix} 1 & \alpha_1 & 0 & 0 & \cdots \\ \beta_1 & 1 & \alpha_2 & 0 & \cdots \\ 0 & \beta_2 & 1 & \alpha_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

is that the series $(2)\Sigma\alpha_n\beta_n$ shall be absolutely convergent. Then by use of Poincaré's theorem (1886) the author deduces that the convergence of the determinant (1) depends on the convergence of the simpler series $(3)\Sigma(|\alpha_n|+|\beta_n|)$. In the latter case, however, the condition is only sufficient. This is proved and illustrated by a special example. As an extension to the above lemma, the author gives proofs of convergence of more general determinants which arise in his earlier paper on *Solutions of homogeneous linear difference equations by means of infinite determinants* (MS at University of California Library, Berkeley). In the proofs of the "extended" determinants are shown first the necessary and sufficient conditions of convergence of those determinants. Then by applying Poincaré's theorem again, it is deduced that the absolute convergence of those determinants depends on the absolute convergence of certain series of linear terms. Here again the condition is only sufficient as is shown by constructing special examples. (Received October 27, 1930.)

21. Professor Morgan Ward: *A relational definition of an abstract arithmetic.*

This paper gives a definition of the basic property of common arithmetic, unique decomposition into prime factors, in terms of the relation of division. The formulation consists of a set of five postulates which are shown to be consistent and independent. (Received November 7, 1930.)

22. Professor Herman Betz: *Periodic motions and surface transformations.*

This paper continues in detail the investigation into the stability of the periodic orbits started in a previous paper whose abstract appeared in this Bulletin. It seeks to establish the stability by means of extensive calculations. (Received November 13, 1930.)

23. Dr. D. G. Bourgin: *Characterization of the sine and cosine by their orthogonality properties.*

If (I), $f(x)$ is summable and of summable square in the sense of Lebesgue and if (II), the functions $f(nx)$ with n an integer form a normalized orthogonal set on the interval $-\pi$, π and if (III), $f(x)$ is an even (odd) function then $f(x)$ is $\cos kx$ ($\sin kx$)(k an integer $\neq 0$), except on a point set of measure zero. Quite recently Titchmarsh has surmised (without proof) a somewhat similar theorem, though his statement is invalid due to omission of a condition analogous to the necessary condition III. Condition (I) is dictated by the nature of the proof. A like theorem probably holds under less restrictive conditions. (Received November 20, 1930.

24. Professor J. S. Turner: *The cross ratio of the points in which a transversal cuts the sides of a quadrilateral.*

Let a transversal cut the sides AB, BC, CD, DA of a quadrilateral in A_1, B_1, C_1, D_1, respectively, and let p_1, p_2, p_3, p_4 be the perpendiculars from A, B, C, D on A_1B_1; then $(A_1\ B_1\ C_1\ D_1) = k\ p_2p_4/(p_1p_3)$, where k is a constant. As an application it is shown that if A_1B_1 touches a conic inscribed in $ABCD$, then $p_2p_4/(p_1p_2)$ is constant. (Received November 11, 1930.)

25. Dr. Wilhelm Maier: *Orthogonal properties of elliptic functions.* Preliminary report.

In this paper a function $s(x, y; u)$ is defined; a representation of the odd Jacobian ϑ function is found for complex values of x, y, u; and other identical relations are derived. If we set $s(0, y; u) = s(y; u)$, and if we take an appropriate set of zeros $y_\nu(\nu = 0, \pm 1, \cdots)$ of $\vartheta'(y)$ as "characteristic" values, and the values of $s(y; u)$ as "characteristic" functions, we find elliptic functions to be orthogonal. Another identity is derived for the standard function $p(u; w_1, w_2)$ of Weierstrass, as a result of an identity between the s functions, as soon as we compare couples of orthogonal vectors in three and in n dimensions for $n \rightarrow \infty$. We can therefore develop analytic functions as a series of ϑ functions. With $\partial c_\nu/\partial u = 0$, it follows that $A(u) = \sum_{\nu=-\infty}^{+\infty} c_\nu \vartheta(u + y_\nu)$. If the given function is an integral function, the domain of convergence of the series is found to be a parallelogram. (Received November 12, 1930.)

26.* Dr. J. J. Gergen: *Note on the Green's function of a star-shaped, three-dimensional region.*

* The papers beyond this point have been read, or are to be read, at meetings of which the reports have not yet been published.

The chief result in this note is as follows. Suppose that the region D is star-shaped with regard to a point p and that g is the Green's function of D with pole at p; then the gradient of g vanishes nowhere in D. Furthermore, for every positive constant C, the region where $g>C$ is star-shaped with regard to p. (Received November 8, 1930.)

27. Dr. G. W. Starcher: *On identities arising from solutions of q-difference equations and some interpretations in number theory.*

In this paper certain linear homogeneous q-difference equations, with linear coefficients, and of the first and second order respectively are considered. Unique solutions for each of these are obtained in the form of simple infinite products, infinite series with simple coefficients, or the product of two or more such expressions. Equating different expressions for a given solution gives an identity, and by comparing such identities it is shown that many different, equally simple, expressions may represent the same function. Identities involving infinite series and products which appear to be new are exhibited; many well known identities are obtained, and often new proofs for them are afforded; new members are added to some well known identities; and certain identities are given interpretations in number theory. (Received November 20, 1930.)

28. Professor B. W. Jones: *The regularity of genus of positive ternary quadratic forms.*

In a manner analogous to L. E. Dickson's definition of a *regular* form, a genus of positive ternary quadratic forms is said to be *regular* when the integers represented (or not represented) by the forms of the genus are exclusively those in certain arithmetic progressions. In this paper, by a generalization of Dirichlet's method for particular forms, it is proved that every genus is regular; that is, with every genus is associated a set of arithmetic progressions such that no number in any progression of the set is represented by a form of the genus and that every positive integer not in any progression of the set is represented by some form of the genus. An important application is that if any form represents the only class in the genus, it is regular. (Received November 21, 1930.)

29. Dr. R. P. Agnew (National Research Fellow): *On double orthogonal series.*

An infinite double series of the form $\sum\sum a_{ij}\phi_{ij}$, where $\{a_{ij}\}$ is a real double sequence and $\{\phi_{ij}\}$ is a system of normal orthogonal functions, is called a double orthogonal series. In this paper numerous properties, common to all double orthogonal series of which the coefficients satisfy the relation $\sum\sum a^2_{ij}<\infty$, are developed. Convergence almost everywhere of a certain family of double subsequences of the double sequence of partial sums of such an orthogonal series to a measurable function f of summable square is established. The double series is shown to be the orthogonal development of f in terms of $\{\phi_{ij}\}$, and to converge in the mean to f. Parseval's theorem is extended to double orthogonal series. It is pointed out that the convergence of $\sum\sum a^2_{ij}$ is necessary as well as sufficient for convergence in the mean of a double orthogonal series, and necessary as well as sufficient in order that a double orthogonal series may be the orthogonal development of a function of summable square. A few facts pertaining to convergence almost everywhere of double orthogonal series are given. The author hopes to discuss, in another paper, convergence and summability of double orthogonal series. (Received November 24, 1930.)

30. Professor T. H. Hildebrandt: *On the interchange of limit and Lebesgue integral for a sequence of functions.*

The object of this note is to prove the following theorem: If $U(n,\delta)$ is the least upper bound and $L(a,\delta)$ is the greatest lower bound of $\int_e f_n$ for all measurable subsets e of E for which $me \leqq \delta$, then a necessary and sufficient condition that $\lim_n \int_E f_n = \int_E f$ (where $\lim_n f_n = f$ on E), is that $\lim_{\delta\to 0} \lim_{n\to\infty} [U(n, \delta) + L(n, \delta)] = 0$. (Received November 24, 1930.)

31. Professor M. S. Knebelman: *Multivectorial curvature.*

Riemannian curvature is usually defined for a two-dimensional orientation. This notion is extended to r-dimensional orientations $1 < r \leqq n$, n being the dimensionality of the metric space. It is then shown that if the r-dimensional curvature does not depend on the orientation for any fixed $r < n-1$, the curvature of every dimensionality is constant throughout the space. It is also shown that if the $(n-1)$-dimensional curvature is independent of orientation it is constant and the space is necessarily an Einstein space, the last result serving as a geometric characterization of an Einstein space. The n-dimensional curvature as here defined reduces to the scalar curvature of the space and is always independent of orientation. The above results however need not be true in this case. (Received November 24, 1930.)

32. Professor R. L. Moore: *Spaces in which there exist consecutive points.*

The author has constructed a set of axioms in terms of the undefined notions *point, region* and *"consecutive to."* If the term "consecutive to" is properly interpreted, these axioms are satisfied in certain spaces in which the "points" are the elements of an upper semi-continuous collection of continua that are not all mutually exclusive. (Received November 28, 1930.)

33. Professor E. G. Olds: *A method of classification for the distributions obtained in small samples.* Preliminary report.

If a finite number of objects are classified into groups and a small sample taken, the possible distributions of the sample may be written down and the probability of each computed. In this paper it is pointed out that the distributions may be classified into sets on the basis of the sum of the absolute values of the difference between the number obtained in each group and the number expected. It is then possible to compute the probability of getting some one distribution of any particular set without evaluating the probabilities for the individual members of the set or even finding the number in the set. The paper then gives the results obtained after applying this method to a consideration of the various distributions of a class of twenty students, chosen from one hundred students who were previously divided into six groups. (Received November 25, 1930.)

34. Professor D. V. Widder: *Necessary and sufficient conditions for the representation of a function as a Laplace integral.*

This paper determines necessary and sufficient conditions for the representation of a function $f(x)$ by means of a Stieltjes integral of the form $f(x) = \int_0^\infty e^{-xt}\, d\alpha(t)$. This integral may be regarded as a generalization of Taylor's series. On the other hand the above equation may be regarded as a singular integral equation for the determination of $\alpha(t)$. From this point of view it serves to generalize the moment problem of Hausdorff $\mu_n = \int_0^1 t^n d\alpha(t)$ $(n = 0, 1, 2, \cdots)$. For, if the variable n is replaced by a continuous variable, the latter integral equation reduces to the former. Hausdorff showed that the moment problem has a non-decreasing solution $\alpha(t)$ if and only if the successive differences of the constants μ_n are alternately positive and negative. Likewise the integral equation has a non-decreasing solution $\alpha(t)$ if and only if the successive derivatives of $f(x)$ are alternately positive and negative. Necessary and sufficient conditions on $f(x)$ for the existence of functions $\alpha(t)$ of various types are also treated. For example, we may demand that $\alpha(t)$ should be an integral, when the given equation reduces to $f(x) = \int_0^\infty e^{-xt}\phi(t)dt$. A necessary and sufficient condition for the representation of a function in Dirichlet's series is obtained as an application. (Received November 25, 1930.)

35. Professor P. R. Rider: *Distribution of the correlation coefficient in small samples from non-normal universes.* Preliminary report.

The distribution of the correlation coefficient, r, in samples from a bivariate normal universe has been derived by R. A. Fisher, but little if anything has been done with samples from other types of population. The author has been studying, by means of actual sampling, the distribution of r for samples from non-normal universes. For the particular universe for which results have already been obtained ($z = 1$ for $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$) the standard error of r is considerably greater than for samples from a normal universe. (Received November 25, 1930.)

36. Dr. Leo Zippin (National Research Fellow): *On a problem of N. Aronszajn and an axiom of R. L. Moore.*

N. Aronszajn, in the Fundamenta Mathematicae, vol. 15 (1930), raises the question whether a space satisfying the central axiom of his paper is necessarily a G_0-subset of every metric space of which it is a subspace; this is answered affirmatively. It is further proved that this axiom, for a class of topologic spaces which include the metrizable, is equivalent to a considerably earlier axiom due to R. L. Moore, with which Aronszajn appears not to be acquainted and from which R. L. Moore drew the consequences that form the subject of Aronszajn's work (see abstract, this Bulletin, vol. 33 (1927), p. 141; also, Synopsis of the Colloquium Lectures held at Boulder, August, 1929, where an equivalent axiom is used; these results are part of a general treatment of the field of analysis situs to appear in the regular Colloquium Series). In general, however, the axiom of Aronszajn does not imply the axiom of Moore, although the converse is true. (Received November 25, 1930.)

37. Mr. Jesse Pierce: *Solutions of differential equations in the neighborhood of singular points.*

This paper first considers a system of differential equations in which the coefficients have a pole of order k at $t=0$. The solution is obtained in the form of a power series in log αt, the coefficients of which are Laurent series with a finite number of negative exponents. The coefficients of the respective powers of log αt are independent of α. The α occurring in the solutions is defined as $1/t_0$ where the initial conditions are $x_i(t_0)=a_i$. The paper next considers a system in which the functions involved are analytic in both the dependent variables $x_i(i=1, 2, \cdots, n)$ and the independent variable t in the region D defined by $|x_i-a_i|\leqq r_i$, $0\leqq p_1|t|\leqq p_2$. The solution in this case is again obtained as a power series in log αt, the coefficients of which are Laurent series. The solutions in each case are shown to converge for every value of t interior to the original domain D. The method of the present paper is also very effective in obtaining the solutions of linear differential equations (homogeneous or non-homogeneous). (Received November 26, 1930.)

38. Professor M. A. Basoco: *On the trigonometric expansion of elliptic functions.*

The problem of expressing an elliptic function in terms of infinite sums of trigonometric functions has been treated by Hermite, Briot and Bouquet, and others. In the present paper, by the use of Cauchy's residue theorem applied to a certain integrand first used by Gomes Teixeira (Journal für Mathematik, vol. 122) in another connection, we are able to represent any elliptic function by means of a certain expansion which is valid in an arbitrarily wide, but finite, strip of the complex plane. Our formula contains as special cases certain classical results. An interesting feature of our expansion is that it yields the Fourier series development of the function, quite directly. (Received November 26, 1930.)

39. Professor M. A. Basoco: *On certain theta constants.*

In this note we obtain certain recurrences between the derivatives of the Jacobi theta functions for zero value of the argument. By means of these formulas we may calculate expressions of the form $\vartheta_\alpha^{(n)}/\vartheta_\alpha (n$ even, $\alpha=0, 2, 3)$, $\vartheta_1^{(m)}/\vartheta_1'$, ($m$ odd), in terms of similar quotients involving derivatives of lower order. The results here given have their source in the pseudo-addition theorems of the theta functions as given by Jacobi in vol. 1, p. 510, of his collected works. (Received November 26, 1930.)

40. Professor M. A. Basoco: *On the Fourier series expansions of certain Jacobian elliptic functions.*

By means of a method originally suggested by Liouville in connection with the expansion of a doubly periodic function of the third kind in a Fourier series, the author has obtained complete lists of Fourier expansions for the third and fourth powers of the twelve Jacobian elliptic functions. These expansions are in a form which may be readily arithmetized and used in certain problems in the theory of numbers. (Received November 26, 1930.)

41. Dr. V. W. Adkisson: *Cyclicly connected continuous curves and correspondences.*

If M is a cyclicly connected continuous curve lying on a sphere and containing only a finite number of simple closed curves, then, in general, every continuous (1–1) correspondence of M into itself cannot be extended to the sphere. However, if we consider only those curves for which this is true, we are able to show that either the curve consists of a simple closed curve or three arcs, AXB, AYB, AZB, having only A and B in common, or every sense preserving correspondence of M into itself leaves invariant two points of the sphere and one simple closed curve of M. (Received November 26, 1930.)

42. Professor J. H. Neelley: *Concerning the possibility of certain binary quartics being line sections of the plane quartic curve of genus zero and the complete system of independent forms of two binary quartics.*

Since investigations of the rational plane quartic curve have been dependent upon the covariant forms of two binary quartics, these forms in their most general form are included in this paper. Seven of these forms are quartics and as such may represent four collinear points of the curve. These seven forms have been examined for such possibility. One may not be a line section; others specialize the curve if they are line sections; and two seem to have no special influence on the curve. (Received November 26, 1930.)

43. Professor A. H. Copeland: *A new definition of a Stieltjes integral.*

The definition of the Stieltjes integral given in this paper does not depend upon the subdivision of the interval of integration. It is given in terms of infinite one-dimensional matrices. The method of constructing these matrices is suggested by certain considerations in the theory of probability. By means of an exceedingly simple analysis it is possible to establish the convergence of thess integrals for a very general class of functions. (Received November 26, 1930.)

44. Mr. Rufus Oldenburger: *Compositions of n-way matrices and multilinear forms.*

It is known that n-way matrices may be associated with multilinear forms whose coefficients are elements of these matrices. The author extends the ordinary methods of composition of matrices on files of elements, corresponding to row and column multiplication, to multiplication on sub-matrices (couches) of elements, and shows how this is related to a corresponding composition of multilinear forms. By choosing sub-matrices of different dimensions one obtains various manners of multiplication of two given matrices. Matrix compositions may be also equivalent to a homogeneous transformation on the variables of a form, and also to a scalar product of two forms. An algebra of matrices and a corresponding algebra of forms result where there exist identity matrices for every space and ranges on the directions of this space, and for certain matrices a unique inverse matrix which reproduces an identity matrix in multiplication on the right and its complementary identity matrix in multiplication on the left. Several theorems involving the inverse, transpose, etc., are shown to hold. The concept of couche rank defined by Rice and Hitchcock is used in the reduction and canonization of multilinear forms. (Received November 26, 1930.)

45. Professor C. G. Latimer: *The class numbers of real cyclotomic fields.*

This paper gives a finite expression for the class number h of a real algebraic field F of degree E and a divisor of the field Ω defined by a primitive Mth root of unity, under the restrictions that neither E nor M contains a square factor >1. If every subfield of F of prime degree is a primary divisor of Ω our expression for h reduces to the same simple form which was obtained by Fueter for the case where E and M are odd primes (Journal für Mathematik, vol. 147 (1917), pp. 174–83). Our result overlaps that obtained by Fuchs who treated this problem with the restriction that the group to which F belongs be cyclic (Journal für Mathematik, vol. 65 (1866), pp. 74–111). (Received November 28, 1930.)

46. Professor M. J. Weiss: *The degree of simply transitive primitive substitution groups of class u.*

This paper investigates the degree of primitive substitution groups of class u, and in particular the degree of simply transitive primitive substitution groups of class u. Some new properties of substitutions of degree $u=qp$ and odd prime order p in a transitive group of class u are developed. With the aid of these properties and the known properties due to Jordan and Manning, the maximum degree of a transitive subgroup in a primitive group of class u is found. When the group is a simply transitive primitive group, this degree gives the maximum degree of the group. For $q<p^2$, we may state that the degree of a simply transitive primitive group of class qp which contains a substitution of prime order p and degree qp does not exceed the larger of the two quantities $q^2+2pq-2q$ or $2q^2-2qp-2p^2+q$. For q unrestricted the formulas are more complicated. (Received November 28, 1930.)

47. Dr. W. D. Baten: *Euler-Maclaurin summation formula for two variables by use of Bernoulli polynomials in two variables.*

The first part of this article develops certain polynomials in two independent variables, and several properties of these polynomials. The second part is devoted to deriving the Euler-Maclaurin summation formula for two independent variables by employing a certain periodic function which is defined in terms of the polynomials obtained in the first part. A remainder is given in several different forms, together with conditions for convergence of the series obtained for the summation when the summations extend to infinity. (Received November 28, 1930.)

48. Dr. Leonard Carlitz (National Research Fellow): *Note on diophantine automorphisms.*

As Professor Bell has pointed out (this Bulletin, vol. 33, pp. 71–80) Eisenstein's quaternary quartic—the discriminant of the general binary cubic—is essentially the only non-trivial case of a diophantine automorphism known. In this note a class of forms that include the Eisenstein and related forms are considered and it is shown that (i) each form of the set gives rise to a Cremona transformation of period two; (ii) the Hessian of each form is a power of the form. (i) forms a sort of converse to a result of L. Weisner's (this Bulletin vol. 33, pp. 707–712); (ii) generalizes an observation of Cayley's relative to the Eisenstein form. (Received November 28, 1930.)

49. Dr. Leonard Carlitz (National Research Fellow): *Numerical functions in the ring of polynomials*, mod p.

In this paper are defined a number of functions analogous to the well known functions of ordinary rational arithmetic. It appears that the new functions have simpler properties and can be handled much more easily than the ordinary ones so that the arithmetic of the ring of polynomials is actually simpler than ordinary arithmetic. (Received November 28, 1930.)

50. Professor H. C. Shaub: *Plane cubic Cremona transformations with coincident fundamental points.*

Three types of plane cubic Cremona transformations and their inverses are discussed, as a generalization of the ordinary transformation by means of reciprocal radii. (Received November 28, 1930.)

51. Professor H. P. Robertson: *Groups of motions in spaces with distant parallelism.*

The theory of groups of motions in spaces defined by orthogonal ennuples ("n-Beine") is developed and applied to the work of Einstein and Mayer and to relativistic cosmology. A rather surprising result of these investigations is that although there exist 3-spaces admitting a group of G_6 of motions whose associated metric spaces are of arbitrary constant Riemannian curvature, the only one which is in addition invariant under reflection is euclidean. (Received November 28, 1930.)

52. Mr. W. O. Menge: *Reduction of linear transformations to canonical forms.*

The purpose of the paper is to give a complete treatment of the problem of reducing any matrix (or linear transformation in n variables) to the "rational" and "classical" canonical forms. Dickson in his *Modern Algebraic Theory* establishes the existence of the rational and classical canonical forms by properly choosing certain sets of linear forms in such a way as to form chains of maximum length. The general method given in this paper affords a very simple scheme for obtaining the transformations that reduce a matrix (or linear transformation) to either of the two canonical forms, and the reductions are easily carried out in particular cases. The method is established by introducing matrix equations with the aid of several theorems regarding invariant factors. These theorems include among others a proof that the first invariant factor of any matrix is the expression of lowest degree which vanishes when the variable is replaced by the matrix. The application of these canonical forms is shown by obtaining solutions of systems of linear differential and difference equations. (Received November 28, 1930.)

53. Mr. A. T. Craig: *On the distribution of certain statistics obtained from small random samples.*

In this paper general methods are developed for obtaining the distributions of arithmetic means, geometric means, harmonic means, medians, quartiles, and deciles of samples of n drawn from a very general universe. If $f(x)$, $0 \leqq x \leqq \infty$, is the integrable probability function for x, and x_i $(i=1, 2, \cdots, n)$ are independent values of x, then the probability functions for $y=\sum_{i=1}^{n} x_i$ and $w=\prod_{i=1}^{n} x_i$ are derived; and the distributions of the medians of samples of $n=2m+1$ are found. The distributions of standard deviations of samples of two are also derived. Existing theory on the distribution of means of samples is brought under this general theory and in addition the distributions of means of samples from certain Pearson curves as yet untreated are found. General solutions depend upon the integrability of certain functions. The frequency function for $(x_i x_2 \cdots x_k)/(x_{k+1} \cdots x_n)$, where $x_i (i=1, 2, \cdots, n)$ are uncorrelated and have known frequency laws, is a by-product. (Received November 28, 1930.)

54. Dr. P. M. Swingle: *Biconnected and related sets.*

In this paper a number of theorems are proved concerning biconnected (Fundamenta Mathematicae, vol. 2, pp. 214–216) and related sets. (Received November 28, 1930.)

55. Dr. T. S. Peterson: *Systems of linear integral equations with applications to an invariantive theory of quadratic functional forms in n functions.*

This paper develops systematically the Fredholm theory of systems of linear integral equations. By use of certain compact notations, this study is made possible without reduction to a single linear integral equation. By using the Fredholm subgroup of the above transformations, that is, when $k_j^i(\alpha) = \delta_j^i$ (Kronecker delta), a study of the invariants of the quadratic function form is effected. Other interesting theorems are developed for restricted transformations of the type studied. (Received November 28, 1930.)

56. Dr. W. T. Reid (National Research Fellow): *On boundary value problems associated with double integrals in the calculus of variations.*

Lichtenstein (see Monatshefte für Mathematik und Physik, vol. 28 (1917), pp. 3–51, and Mathematische Zeitschrift, vol. 6 (1920), pp. 26–51) has considered a certain double integral problem with fixed boundary in the calculus of variations and has shown that the necessary condition of Jacobi is closely related to a certain boundary value problem associated with Jacobi's differential equation. If Jacobi's condition is satisfied, then the smallest positive characteristic number λ_1^+ of the boundary value problem which he considers must satisfy the inequality $\lambda_1^+ \geqq 1$, and conversely. To establish a certain minimizing property of λ_1^+ he has made use of some expansion theorems for arbitrary functions in terms of the characteristic solutions of the associated boundary value problem. In the present paper this property is established in a much more elementary manner by methods of the calculus of variations, in particular, the existence of λ_1^+ is established in this manner. It is shown that in the associated boundary value problem the parameter may be allowed to enter in a simpler form than that used by Lichtenstein. Also, sufficient conditions are given for an external surface to render the double integral a weak relative minimum. (Received November 29, 1930.)

57. Professor G. Y. Rainich: *Generalization to three dimensions of the Cauchy integral formula.*

The formula is an immediate generalization of the Cauchy integral formula giving the value of an analytic function in a point surrounded by a contour in terms of its values on the contour. In the generalization three independent variables appear as the arguments and three functions of these variables take the place of the real and imaginary parts of the analytic function. These three functions are assumed to possess total differentials and to satisfy four differential equations which express the fact that the functions are partial derivatives of a harmonic function—the Cauchy-Riemann equations constitute a special case of these. The values of the functions at a point are given as integrals of linear combinations of the functions taken over a closed surface surrounding the point. Existence of higher derivatives and power series expansions follow as in the two-dimensional case. (Received November 29, 1930.)

58. Professor G. T. Whyburn: *Concerning locally connected spaces.*

Let M denote a metric, separable, connected and locally connected space. In this paper it is shown that if K is any closed and compact subset of M and A and B are any two components of K, then there exist two mutually exclusive regions (ie., connected open subsets) R_a and R_b of M containing A and B respectively and such that $R_a+R_b \supset K$ and $F(R_a)=F(R_b)=M-(R_a+R_b)$, where $F(X)$ denotes the boundary of the set X. It is also shown that if M is locally compact, then if R is any compact region in M and N is any closed subset of $M-R$ such that $\overline{R}\cdot N$ is totally disconnected there exists a compact region G such that (1) $\overline{G}\cdot N=\overline{R}\cdot N$, (2) $G+\overline{R}\cdot N$ contains a compact continuous curve H containing $\overline{R}\cdot N$ and such that $H-\overline{R}\cdot N$ is connected and contains R, (3) every point of $\overline{R}\cdot N$ is accessible from $H-\overline{R}\cdot N$ and hence also from G. Both of these results have been found useful in studying the structure of certain kinds of continuous curves. (Received November 28, 1930.)

59. Professor J. S. Taylor: *Lebesgue derivatives.*

This paper concerns itself with definitions of Lebesgue derivatives over an interval and at a point. The Lebesgue derivative of $y=f(x)$ over an interval (a, b) is defined as the limit of the ratio of a change in y to the change in the measure of the set of x's in the interval for which the function is less than y as the change in y approaches zero as a limit. Among the definitions of a Lebesgue derivative at a point is that obtained by letting the interval (a, b) shrink to a point. Left-hand and right-hand derivatives are defined in an obvious manner, with upper and lower derivatives obtained in a somewhat similar fashion. The nature of some of the other definitions presented may be indicated briefly by noting that the concept of measure may be applied to the change in y instead of to the change in x, or to both. In all cases the Lebesgue derivative is indentical with the ordinary derivative if the latter exists. The geometric significance of Lebesgue derivatives is studied in detail and certain physical applications are indicated. (Received November 29, 1930.)

60. Professor H. B. Curry: *Apparent variables from the standpoint of combinatory logic.*

This paper is a continuation of an earlier paper entitled *The universal quantifier in combinatory logic* (this Bulletin, abstract 36-3-125). Its purpose is to carry the development far enough so that the notation of apparent variables may be used in subsequent investigations. Expressions involving apparent variables are first defined, in the sense that a process is given whereby a uniquely determined entity is associated with each such expression. It is then shown explicitly that the inferences, which are ordinarily made by means of the apparent variables, may be justified on the basis of the combinatory primitive frame. The treatment is, however, confined to universal apparent variables. Several propositions of the sort just described, and also some propositions regarding formal implication, are proved as formulas; these require some new axioms. (Received November 29, 1930.)

61. Dr. Jacob Yerushalmy (National Research Fellow): *Construction of pencils of equianharmonic cubics.*

Chisini (Rendiconti del Circolo Matematico di Palermo, vol. 41) gives the construction of pencils of cubics of equal generic modulus. In the equianharmonic cases he only shows that to such a pencil corresponds a pencil of lines cutting a quartic curve f_4 in equianharmonic quadruples of points, and that the equation of f_4 is $(\partial^2\phi_3/\partial x_3^2)\phi_3-(1/2)\ (\partial\phi_3/\partial x_3)^2=0$, where ϕ_3 is an arbitrary cubic. Studying the cubic surface which admits f_4 as branch curve and then mapping it on a plane, we obtain the construction of the pencils. We show that a pencil of equianharmonic cubics is contained in a net of such cubics through six base-points. These base-points form the vertices of two incircumscribed triangles in a general cubic curve which are threefold perspective from the vertices of a third incircumscribed triangle in the same cubic. (Received November 29, 1930.)

62. Professor A. D. Michal: *One-parameter linear functional groups in several functions of two variables.*

This paper is concerned with one-parameter continuous groups defined by the integro-differential equations $\partial^i z_T/\partial_T={}_i{}^iH^{*j}z_T$ $(i=1, 2, \cdots, n)$, where the symbol * denotes composition of the first kind in the Volterra sense. Certain integral addition properties of the Bessel functions $J_m(w)$ are considered and are derived from the properties of generalized Volterra transcendentals. This paper will appear in an early issue of this Bulletin. (Received November 29, 1930.)

63. Professor A. D. Michal and Mr. R. S. Martin: *The introduction of Stieltjes integrals in the transformation theory of functional forms.*

Single and multiple Stieltjes integrals are introduced in the study of functional forms and their transformation properties. Some theorems, believed to be new, on Stieltjes integrals are proved and then applied to the functional transformations considered in the earlier work on the subject. (Received November 29, 1930.)

64. Mr. R. S. Martin: *Note on functional forms quadratic in a function and its first derivatives.*

This paper proves the theorem (a generalization of a theorem of A. D. Michal and L. S. Kennison) that a normal quadratic functional form in a function $y(x)$ and its first p derivatives with continuous coefficients can be reduced to a form with continuous coefficients on the composite range $z(x)$, w^1, w^2, $\cdots$, w^p. This paper will appear in an early issue of this Bulletin. (Received November 29, 1930.)

65. Professor Edward Kasner: *Isogonal trajectories in space.*

The ∞^2 isogonal trajectories of any family of ∞^1 curves in a plane have the well known Cesàro-Scheffers property (1900) that the circles of curvature at any point P meet at a second point P'. The characteristic property of the transformation from P to P' was given by the present writer in connection with his study of natural families (Transactions of this Society, 1909, Princeton Colloquium, 1913). The situation in space for isogonals, as contrasted in the natural families, is entirely different. Any congruence of ∞^2 curves has ∞^∞ isogonal trajectories. It is shown that these have the Meusnier property that the circles of curvature lie on a sphere. This follows from the equation $Ay''+Bz''+C=0$ (see this Bulletin, 1908). If any two congruences of curves in space are given there exist ∞^4 curves isogonal to both. The rather complicated properties of the circles of curvature at a point are studied. For a single congruence there exist ∞^4 isogonals of least curvature; the centers of curvature for a given point generate an algebraic twisted curve of high order. (Received November 30, 1930.)

66. Professor Edward Kasner: *Near-collineations and Blaschke's Textilemathematik.*

A near collineation, in the case of the plane, is defined as a point transformation which converts $3\infty^1$ straight lines into straight lines (see previous abstract and papers in this Bulletin, 1903, and Transactions of this Society, 1905). Blaschke, in his recent geometry of textiles, uses the term *Gewebe* for any system of $3\infty^1$ curves, and defines a *Sechsergewebe* as the special case equivalent topologically to three sets of parallel straight lines. The fact that Gewebe (as contrasted with nets of $2\infty^1$ curves) have invariants under the group of all (analytic) point transformations was indicated by the writer in his St. Louis address (*The present problems of geometry*, this Bulletin, 1905). In the present paper he studies transformations which convert three pencils of straights into straights. It is shown that the tangents to a curve of third class constitute a Sechsergewebe, the requisite transformation being a general quadratic (non-Cremona) transformation. This includes, as special cases, a pencil and a conic, and any three pencils. All infinitesimal near-collineations which preserve a Sechsergewebe of straight lines are determined explicitly. The topological invariants of four or more pencils are identical with the projective invariants as stated in the St. Louis address. Only very special straight Gewebe admit non-projective transformations into straight Gewebe. (Received November 29, 1930.)

67. Professor T. A. Pierce: *Parametric solutions of certain diophantine equations.*

A generalization of Lagrange's norm method of solving diophantine equations is given. The equations considered here are obtained by equating the norms of integers in two different fields. The method of obtaining parametric solutions is derived from an equation involving the determinants of matrices. (Received November 30, 1930.)

68. Professor R. E. Gilman: *On the Hadamard determinant theorem and orthogonal determinants.*

It is well known that the Hadamard upper bound for the value of a determinant whose elements do not exceed unity in absolute value can, in the case of determinants with real elements, only be attained when the order is a multiple of four. The problem here attacked was proposed by Hadamard in his original paper, namely to discover whether maximum determinants for all orders which are multiples of four attain the Hadamard bound. It is proved that the Hadamard bound is attained for determinants of order $p=2^{\nu}n_1^{\nu_1}n_2^{\nu_2}n_3^{\nu_3}\cdots n_k^{\nu_k}$ where the n_i are multiples of four for which $n_i-1$are prime, and where ν_i-1 is any positive integer or zero. No statement is made as to the other cases. What is believed to be a new proof of the Hadamard theorem for determinants with real elements, is given. (Received November 30, 1930.)

69. Professor R. E. Gilman: *The analog of periodicity for general Fourier series.*

The Fourier series $\sum_{n=1}^{\infty} a_n \sin \beta_n x$, where β_n is a root of the transcendental equation $p \tan u+u=0$, satisfies the integro-difference equation

$$f(x+1) = -f(x-1) + 2p\int_0^{xq(r-x)} f(t-1)dt. \tag{1}$$

This gives the continuation of the series outside the original interval of definition, and is the precise analog of the equation

$$f(x+1) = f(x-1) \tag{2}$$

for the series $\sum_{n=0}^{\infty} a_n \sin n\pi x$. As $p\rightarrow\infty$, (1) actually approaches (2) as a limit. (Received November 30, 1930.)

70. Professor G. C. Evans: *A new type of Cauchy integral.*

A Cauchy integral $w(z)=w(M)$ is taken around a simple closed regular curve C without vertices, the numerator being in the form $\xi(P)+i\eta(P)$, where ξ and η are real-valued functions summable on C in the Lebesgue sense. A necessary and sufficient condition that $w(M)$ should take on "in the wide sense," at points Q of a set of positive measure on C, the boundary condition

$$\lim_{M=Q} W(M) = \xi(Q) + i\eta(Q) - (1/\pi)\int_C[\xi(P) + i\eta(P)] \cos(QP, n_p) \cdot ds_p/RP + \text{const.},$$

is that the function $w_2(z)$ obtained by replacing $\xi(P)+i\eta(P)$ in $w(z)$ by $\xi(P)-i\eta(P)$ should be identically a constant within C. The degree of generality of this representation is shown by the fact that also a necessary and sufficient condition for it is that the potential of a double layer $W(M)=-(1/\pi)\int_C[\xi(P)+i\eta(P)]\cos(MP, n_p)ds_p/MP$ should be holomorphic within C—a condition which includes all the usual cases of representation as a Cauchy integral of the customary form. We have $W(M)=w(M)+$const. Finally, if the representation as Cauchy integral with the above boundary condition is possible, then $w(z)$ may also be written as a Cauchy integral around C with numerator$f(P)+ig(P)$ where f and g are real-valued functions summable on C in the Lebesgue sense, and where $w(M)$ takes on, "in the wide sense," at almost all points Q of C, the boundary values $f(Q)+ig(Q)$. (Received November 30, 1930.)

71. Dr. W. J. Trjitzinsky (National Research Fellow): *Some general theorems on composition of singularities.*

Among the results of this paper the author proves a general theorem on composition of singularities applicable to non-uniform functions. In particular, a theorem is given which constitutes a synthesis of the two classical theorems on composition of singularities, due to Hadamard and Hurwitz. (Received December 1, 1930.)

72. Professor J. Geronimus: *On orthogonal polynomials.*

In this paper it is shown that a set of polynomials orthogonal and normal on the interval $(-1, 1)$ with a characteristic function satisfying suitable conditions will be orthogonal and normal on a contour in the complex plane with a certain other characteristic function simply expressible in terms of the first. A number of identities and expansions in series are then obtained involving the orthogonal polynomials and the associated "functions of the second kind." (Received November 20, 1930.)

73. Dr. J. H. C. Whitehead: *On linear connections.*

Linear connections of the type with which this paper deals were first studied by R. König (Jahresbericht der Vereinigung, 1920), who considered an n-dimensional manifold, with each point of which he associated a linear space of m dimensions. The linear spaces at different points of a curve are related by means of linear differential equations, which involve a linear connection and which are analogous to those defining parallel displacement in the affine theory. B. V. Williams and the author (Annals of Mathematics, 1930) showed how to construct an integrable connection which agrees with a given connection on a given series of sub-spaces. This paper gives a descriptive proof of this theorem, and uses similar methods in constructing a family of coordinate systems in terms of invariant figures (paths and parallel vectors) determined by a given affine connection. The foundations are laid for a systematic study of a linear connection together with an affine connection. A theorem is proved about a linear connection by itself which also holds for symmetric affine connections, and gives a simple expression for the alternating sum of the components of the repeated covariant derivatives of certain tensors. Finally it is shown how the theorem proved by Williams and the author is relevant to Cartan's theory of Pfaffian equations, when the latter is applied to the equations for linear displacement. (Received November 19, 1930.)

74. Professor J. L. Walsh: *The existence of rational functions of best approximation.*

This note shows the existence of a rational function (of the complex variable) of degree n of best approximation in the sense of Tchebycheff to an arbitrary continuous function on an arbitrary closed point set which is dense in itself. The proof of this existence is an application of the following theorem, proved by the theory of normal families of functions: If the moduli of an infinite family of rational functions all of degree n are uniformly limited in $2n+1$ points of the extended plane, then there can be extracted from the family an infinite subsequence converging continuously in the extended plane with the omission of n points. The limit of this sequence is a rational function of degree n. (Received November 26, 1930.)

75. Mr. W. W. Flexner: *The Poincaré duality theorem for topological manifolds.*

A topological manifold is a compact separable space whose defining neighborhoods are combinatorial *n-cells.* This paper extends to topological manifolds the major results established for manifolds that are ordinary complexes: the invariance of the homology characters, the properties of the Kronecker index and above all the Poincaré duality theorem with its extensions by Veblen and Alexander. A preliminary report by Lefschetz and Flexner is in the Proceedings of the National Academy of Sciences, vol. 16 (1930), No. 8, and the complete papers will appear in the Annals of Mathematics. (Received December 1, 1930.)

76. Professor Lincoln La Paz: *Note on variation problems with prescribed transversality coefficients and extremals.*

If $T_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$, $(i=1, \cdots, n)$, is an admissible set of transversality coefficients, the most general integrand function, f, of a problem of minimizing $I=\int_{x_1}^{x_2} f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')\,dx$ for which the transversality coefficients are T_i has been determined by the writer and is found to contain an arbitrary function $g(x, y_1, \cdots, y_n)$ as a factor (see this Bulletin, vol. 36, p. 674). In this note it is shown that when in addition an admissible $2n$-parameter family of curves is prescribed as the extremal system of I the function g is determined up to a constant factor. This result is obtained without requiring that the prescribed system of curves be defined by equations of the form $\psi_i(x, y_1, \cdots, y_n, a_1, \cdots, a_n)=c_i$ $(i=1, \cdots, n)$. (Received December 1, 1930.)

77. Professor G. T. Whyburn: *Concerning the sum of a monotonic increasing sequence of continuous curves.*

In this paper it is shown that if $M_1, M_2, \cdots$ is any monotonic increasing sequence of continuous curves and there exists a convergent series of positive numbers $a_1+a_2+\cdots$ such that, for each n, every point x of M_{n+1} can be joined to some point y of M_n by a connected subset of M_{n+1} of diameter $<a_n$, then if S denotes the set $M_1+M_2+\cdots$, $\overline{S}$ is a continuous curve. Furthermore, each point of $\overline{S}-S$ is regularly accessible from S. This result is applied in proving the following theorem: Let R be any connected open subset of a continuous curve M and let K be any closed and compact subset of the boundary of R in M each point of which is regularly accessible from R; then there exists a continuous curve H in $R+K$ which contains K and is such that $H-K$ is connected and every point of K is regularly accessible from $H-K$. Also a condition is given under which the product of a monotonic decreasing sequence of continuous curves will be a continuous curve. (Received December 1, 1930.)

78. Professor Tobias Dantzig: *A system of circles, associated with a conformal transformation.*

In previous papers presented before the Society, the author dealt with what he called the indicating conics of a transformation. In the case of a conformal transformation, the indicating conic is a circle. There is thus associated with a conformal transformation T a double infinity of circles (C). Given a system of circles (C) depending on two parameters, the question arises whether a group of transformations may exist that will admit (C) for its indicating system. The author gives here a necessary and sufficient condition that such a group may exist, and determines groups of conformal transformations corresponding to some simple systems of circles. (Received December 1, 1930.)

79. Professor Arnold Dresden: *A composition of determinants.*

In connection with a study of the invariance of multiple integrals of a general type under parameter transformations, the author was led to a consideration of determinants of order n^2, n^3, etc., formed by a process of composition from two, three, or more determinants of order n. The composition here referred to can be best described for the case of two determinants by the statement that each element of one of them is replaced by the matrix resulting when the matrix of the other determinant is multiplied by the element in question. The paper obtains simple formulas for the values of such composite determinants and also for the values of their minors. The process is extended in various directions. These composite determinants are special cases of those considered by E. H. Moore, (Annals of Mathematics, (2), Vol. 1). The formulas obtained for the minors are believed to be new. The approach and treatment differ from those of Professor Moore's paper. (Received December 1, 1930.)

80. Dr. T. C. Benton: *On conical accessibility in E_3 (three-dimensional euclidean space).*

A point which can be *ϵ-separated* by a set of acyclic continuous curves is called an edge point of a continuous surface in E_3. If in the neighborhood of a point x there is a simple closed curve R_x lying on the continuous surface so that $S-R_x=I(R_x)+O(R_x)$ where $I(R_x)>x$ and is homeomorphic to a plane circle plus its interior, then S is flat at x. Conical accessibility was defined by the author in abstract 36-3-189. A continuous surface S which is flat at every point, contains no edge points, and which is conically accessible from any unknotted simple closed curve in E_3-S is a set homeomorphic with a sphere. (Received December 1, 1930.)

81. Dr. T. C. Benton: *Continuous curves which are homogeneous except for a countable infinity of points.*

This problem, a continuation of the author's paper *Continuous curves homogeneous except for a finite number of points* (Fundamenta Mathematicae vols. 13 and 15), falls into two main parts depending on whether the homogeneous points are cut points or non-cut points. In the first case the resulting set must consist of an infinite set of arcs all of which have one point in common and only a finite number of which are of diameter greater than any assigned positive number, however small. In the second case the classification of sets depends on the separation properties of the set C' of limit points, and the set of points which are not limit points, in the set of non-homogeneous points C. The case where points of C' are non-cut points and those of $C-C'$ are cut points has been discussed, and sets of this type fall under two very different types. (Received December 1, 1930.)

82. Professor C. M. Cramlet: *Complete sets of tensors and invariants of systems of linear homogeneous differential equations of the second order.*

The system $(d^2y^i/dx^2)+\sum_{\alpha=1}^k p_\alpha^i(dy^\alpha/dx)+\sum_{\alpha=1}^k q_\alpha^i y^\alpha=0$ is subjected to the most general transformations leaving it linear and homogeneous: $y^i=\sum a_\alpha^i(x)\bar{y}_\alpha$, $\bar{x}=\xi(x)$, the functions of x being arbitrary. When the components of a tensor are multiplied, under a transformation, by the kth power of $d\xi/dx$ the tensor is said to be relative. A mixed relative tensor $\pi_s^r(q_r^s, p_r^s, dp_r^s/dx)$ is discovered in terms of which the equivalence of two basic sets of differential equations can be expressed. A method, called tensor differentiation, of discovering new tensors is formulated and a tensor calculus developed. Complete systems of invariants are obtained by the ordinary tensor methods of outer and summed multiplication. The set of invariants so found are equivalent to the set of irrational invariants found as roots of the characteristic equation det $(\pi_s^r-\lambda\delta_s^r)=0$. These invariants determine unique directions and, in the case of multiple roots, unique subspaces, associated with the points of an integral curve. The integral curves in these directions have special properties. (Received December 1, 1930.)

83. Professor Jesse Douglas: *Solution of the problem of Plateau for three contours.*

In this paper the author solves completely the problem specified in the title by the methods of his paper, *A general formulation of the problem of Plateau* (this Bulletin, abstract 36-1-16). The topology of the minimal surface is supposed to be that of a sphere with three perforations. This is the first paper written on the subject. (Received December 3, 1930.)

84. Professor Jesse Douglas: *Proof of a surmisal of Hilbert concerning the rôle of the Dirichlet principle in analysis.*

In Hilbert's fundamental memoir *Wesen und Ziele einer Analysis der unendlichvielen unabhängigen Variabeln*, Rendiconti di Palermo, vol. 27 (1909), pp. 59–74, the following statement is made (p. 61, eighth line from bottom): "Vor Allem aber gilt die Tatsache, dass eine stetige Funktion unendlichvieler Variabler stets ein Minimum besitzen muss—ein Satz, der uns wegen seiner Präzision, seiner Allgemeinheit und seiner Anwendbarkeit als ein Ersatz für das bekannte Dirichlet'sche Prinzip erscheint." From the results of Parts III and IV of the writer's paper, *Solution of the problem of Plateau*, appearing in the January 1931 number of the Transactions of this Society, it is to be seen that the Dirichlet principle, insofar as it concerns continuous distributions of assigned values on any Jordan curve (a denumerable set of discontinuities in the distribution may be allowed), is a consequence of the following theorem: The quadratic form in a denumerable infinity of variables, $Q(x)=\sum_{p=1}^{\infty} x_p^2$, attains its minimum value on any closed set of points in Hilbert space (consisting of all points x for which $Q(x)$ is finite). (Received December 3, 1930.)

85. Professor Jesse Douglas: *Minimax and instability configurations of a soap film bounded by two contours.*

In this paper it is proved that whenever it is possible to span a stable soap film between two contours there exists a second minimal surface spanned between the two contours, which corresponds to a minimax of the fundamental functional introduced by the writer into this problem, also to a minimax of area, and therefore cannot be realized as a stable soap film. More generally, the possible soap films bounded by two given contours may be arranged in series consisting alternately of stable and virtual (minimax) films. If the two contours undergo continuous variation both as to their form and their position in space, the stable and virtual films vary together until a stable film comes into coincidence with the next virtual film, whereupon it passes over suddenly into the next stable film of its series, until finally, when the contours are sufficiently far apart, there remains only a disconnected film determined by the contours taken separately. The classic case of two co-axial circles, where the virtual film is a catenoid whose meridian contains the conjugate point of its extremity, is an illustration of this general theory. (Received December 3, 1930.)

86. Professor Jesse Douglas: *Example of a Jordan curve in space in which no finite area can be spanned.*

The following example was constructed by the author with the collaboration of P. Franklin. Consider a broken line, broken at right angles and in the form of a spiral, whose successive segments have the lengths $1, 1/\sqrt{2}, 1/\sqrt{3}, \cdots$. On each segment construct a square lying towards the outside of the spiral. In each square let there be a Peano curve starting at the initial point and ending at the terminal point of the corresponding segment. Let the unit interval $0 \leqq t \leqq 1$ be divided by the points $1/2, 2/3, 3/4, \cdots$ into a denumerable infinity of sub-intervals, and let the nth Peano curve be represented as continuous image of the nth interval. If to the equations $x = \phi(t)$, $y = \psi(t)$ representing all these Peano curves laid end to end as continuous image of the unit t-interval (end-point $t = 1$ included), we adjoin $z = t$, then we have a Jordan arc in space. Four of these Jordan arcs give the desired example. The proof results from the fact that the content of the orthogonal projection of the Jordan arc on the xy-plane, counting each point with its proper multiplicity, is the sum of the harmonic series. A fortiori, the content of the orthogonal projection of any surface spanned within the curve is $+\infty$, and this is, a fortiori, the area of the surface. (Received December 3, 1930.)

87. Professor Norbert Wiener: *The closure of the set of translations of a given function.*

The present paper deduces many Tauberian theorems from the fact that a necessary and sufficient condition for the set of translations of a function of L_1 to be closed L_1 is that its Fourier transform should have no zeros. (Received December 3, 1930.)

88. Professor Marston Morse: *A reduction of sufficient conditions in the fixed end-point problem of Lagrange.*

Let g be the given extremal and (ab) the corresponding interval for x. We are concerned with sufficient conditions. Among the customary assumptions is that of normalcy on every subinterval of an interval including (ab) in its interior. It would be more natural to assume merely that g is normal on every subinterval of (ab), and in fact a set of sufficient conditions have been stated by Bliss in which this assumption appears. However, no proof has previously been given. In fact the customary method of proof is inadequate because the family of extremals through a point p on g's extension may be highly improper near p if the normalcy hypothesis is confined to (ab). The author proves the appropriate theorem. Incidentally the so called "oscillation theorem" is proved by simpler methods. A second and new oscillation theorem is proved which makes a Mayer field available. This paper will appear in the Annals of Mathematics. Its results are used by the author in another paper giving for the first time sufficient conditions under the most general end conditions. (Received November 29, 1930.)

89. Mr. L. J. Paradiso: *Linear differential equations whose coefficients are of bounded variation.*

M. Bôcher developed the theory of the non-homogeneous linear differential equation of the nth order where the coefficients may possess a finite number of finite or infinite discontinuities and are absolutely integrable in the sense of Cauchy's extension of the Riemann integral. In this paper an existence theorem for this differential equation is given when the coefficients are of bounded variation and may have an enumerable infinity of points of finite discontinuity. The differential equation is transformed into a functional equation of the type $\phi(x) = f(x) + \int_a^b K(x, s) d\phi(s)$ where $K(x, s)$ is a function of the coefficients in the differential equation. From this solution $\phi(x)$ of the functional equation we can easily obtain the solution of the differential equation. (Received December 6, 1930.)

90. Dr. T. H. Rawles: *On hypergeometric functions of two variables.*

A modification of Pochhammer's integral $\int (u-a)^{\kappa-1} (u-x)^{\lambda-1} (u-y)^{\mu-1} du$ has been suggested as a basis for extending the idea of hypergeometric functions of two variables. Another viewpoint which has been put forward is to define such a function by means of difference equations between the coefficients in the power series development of the function. One of the results contained in this paper is to show that the coefficients of the power series development of the function defined by Pochhammer's integral satisfy a difference equation of a certain type. (Received December 6, 1930.)

91. Mr. H. F. Bohnenblust and Professor Einar Hille: *On the absolute convergence problem for Dirichlet's series.* Preliminary communication.

The authors have extended some recent work of Littlewood on bilinear forms (Quarterly Journal, Oxford series, vol. 1 (1930), pp. 164–174) to m-linear forms. Their extension includes the following theorem: If σ_u is the abscissa of uniform convergence of the Dirichlet's series $\sum a_n n^{-s}$, then the series obtained by omitting all those values of n which contain more than m prime factors is absolutely convergent for $\sigma > \sigma_u + (m-1)/(2m)$. For $m=1$ respectively $m \to \infty$ this becomes a fact proved by H. Bohr (Göttinger Nachrichten (1913), pp. 441–488). For $m=2$ it is the best possible theorem by virtue of a result due to Toeplitz (ibid., pp. 417–432); conversely this work verifies Toeplitz's guess that his solution cannot be improved by bilinear forms. (Received December 12, 1930.)

92. Dr. W. Seidel: *On the cluster values of schlicht and bounded analytic functions.*

Consider a function $f(z)$ schlicht and analytic in the unit circle $|z|<1$. Let the point $z=1$ be a point of discontinuity of $f(z)$ and let $z_1, z_2, \cdots$ be a sequence of interior points of $|z|<1$ converging toward $z=1$ and for which $\lim_{n\to\infty} f(z_n) = \alpha$ exists. We shall call α a *cluster value* of $f(z)$ in the point $z=1$ and the set of the values α obtained by considering all sequences $\{z_n\}$ with the above properties the cluster set of $f(z)$ in the point $z=1$. This paper deals with two questions. First, all cluster values of $f(z)$ obtained by approaching $z=1$ along a Jordan arc are considered and the relation between such a curve and the corresponding cluster set is investigated. Secondly, the author investigates sufficient conditions that two cluster values corresponding to two sequences $\{z_n\}$ and $\{\zeta_n\}$ be equal. The case of harmonic functions and of bounded analytic functions will also be briefly considered. (Received December 18, 1930.)

93. Dr. W. Seidel: *On the correspondence of boundaries in conformal maps.*

Let $w=f(z)$ be an analytic function defined in the unit circle $|z|<1$ and mapping the circle conformally on a region of the w-plane bounded by a single closed Jordan curve C. It is known that in this case the function $f(z)$ will be continuous in the closed circle $|z|<1$. It has also been known for several special cases that if the boundary curve C has some further properties, such as the existence of a tangent in a point, then the mapping function or its first derivative will possess additional properties on the boundary of the unit circle. The author studies these problems systematically and obtains various results about the boundary values of the first derivative and higher derivatives of the mapping function. (Received December 18, 1930.)

94. Dr. I. Schoenberg (International Research Fellow): *The minimizing properties of geodesic arcs with conjugate end points.*

Let E_{01} be an extremal arc for the integral $\mathfrak{J} = \int f(x, y, y')dx$, whose end points 0 and 1 are conjugate. The question whether or not E_{01} actually minimizes the integral $\mathfrak{J}$ has been discussed by A. Kneser, W. F. Osgood, J. W.

Lindeberg and H. Hahn, by studying the shape of the envelope of the extremals through the point 0. Quite another method has been used by L. Lichtenstein. Using the equation of Jacobi, the author gives a method of carrying through the criteria obtained by Osgood and Lindeberg. In the case of a geodesic arc g_{01} on a surface with conjugate end points, the general method of this paper leads to suitable criteria for the minimizing properties of g_{01} in terms of the integral η of the equation of Jacobi-Bonnet $d^2\eta/ds^2+K(s)\eta=0$, which vanishes at both end points of the geodesic arc. (Received December 19, 1930.)

95. Mr. H. F. Bohnenblust and Professor Einar Hille: *On the absolute convergence problem for Dirichlet's series.* II.

Since their preliminary communication (abstract 37-A1-91) the authors have succeeded in showing that the theorem there announced is for every positive integral m the best possible. They have in fact constructed for each m a Dirichlet's series $\sum a_n n^{-s}$ with $a_n=0$ when n is not the product of m primes, which series is uniformly but not absolutely convergent in a strip of width $(m-1)/(2m)$. This proves that Bohr's inequality $\sigma_u-\sigma_a \leqq 1/2$ cannot be improved. (Received December 19, 1930.)

96. Mr. C. J. A. Evelyn and Mr. E. H. Linfoot: *On a problem in the additive theory of numbers.*

The problem considered is that of the representation of positive integers as sums of quadratfrei numbers or, more generally, of "M-numbers" not divisible by any kth power greater than 1, the index k being chosen once and for all. Using the methods of analytic function theory, the authors obtained (Mathematische Zeitschrift vol. 30 (1929), pp. 433-448) asymptotic formulas for the number $v_s(n)$ of representations of a large integer as the sum of s "M-numbers." They are now able (Mathematische Zeitschrift (1931), Journal für die reine und angewandte Mathematik (1931), unpublished) to obtain such formulas by purely arithmetical arguments. (Received December 24, 1930.)

97. Professor G. T. Whyburn: *On the relative cyclic-element decomposition of subcontinua of a continuous curve.*

Let N be any subcontinuum of a continuous curve M with cyclic elements C. A study is made of the decomposition of N into relative cyclic elements given by $(C\cdot N)$. In particular cyclically extendable and cyclically reducible properties of N in this sense are investigated, i.e., properties P such that "P in $(C\cdot N)$" implies "P in N" and conversely. In this connection the properties U_n of Knaster (Comptes Rendus du I Congrès des Mathématiciens des pays Slaves, 1929, p. 288) are studied. It is observed that a continuum may have property U_2 vacuously and not have U_1. Thus U_n does not necessarily imply U_{n-1}. However, if for the given continuum N the property "U_n non-vacuous" implies U_{n-1}, then property U_n is cyclically extendable and reducible in N. Also if H and K are continua with just one common point, then $H+K$ is a U_n-continuum; and conversely, if C is a U_n-continuum and $C=H+K$, where H and K are continua with just one common point, then H and K are U_n-continua. (Received December 24, 1930.)

98. Professor G. T. Whyburn: *Concerning the "closed components" of the non-regular parts of a continuum.*

In this paper the "closed components" of the non-regular parts of a continuum M relative to a given class N of subsets of M (see Hurewicz, Mathematische Annalen, vol. 96, p. 751) are defined by means of separation properties. The new definition is such that the principal theorems concerning the closed components are valid for a more inclusive class of sets N than the normal family (Normalbereiche) of Hurewicz. Thus by particularizing the class N, a number of interesting new results are obtained. Also the following general theorem is proved: If E is any metric, separable and connected space and S is any class of closed subsets of E, there exists a countable subclass $[S_i]$ of S such that if p and q are any two points which may be separated by some set of S, then p and q may be separated by some set of the class $[S_i]$. This theorem also yields many interesting particular theorems, e.g., in any space E (as above) there exists a countable subset D such that every pair of points which may be separated by some countable set may be separated also by some subset of D. (Received December 24, 1930.)

99. Professor G. T. Whyburn: *Concerning the structure of perfect continuous curves.*

In this paper it is proved that any continuum every subcontinuum of which is locally connected (i.e., the continua called perfect continuous curves by R. L. Wilder) is rational in the sense of Menger-Urysohn, i.e., each point is contained in arbitrarily small neighborhoods with countable boundaries. In establishing this result use is made of the following new properties, each of which characterizes perfect continuous curves M among the compact continua: (1) the components and the quasi-components of any subset of M are identical; (2) if p is any point of a component C of any subset K of M and R is any neighborhood of p, there exists a neighborhood V of p such that $R \cdot C \subset V \subset R$ and $F(V) \cdot K \subset F(R) \cdot C$; (3) the components of any subset form a null-family, i.e., at most a finite number are of diameter $> \epsilon$; (4) every subset K of M which is covered by any collection G of open sets can be covered by a null-sequence $V_1, V_2, \cdots$, of mutually exclusive open sets, each being a subset of some element of G and its boundary a subset of the sum of the boundaries of all the elements of G. (Received December 24, 1930.)

100. Professor G. T. Whyburn: *On the cyclic and higher connectivity of locally connected spaces.*

Let C denote any metric, separable, connected and locally connected space which has no cut point. In this paper the following facts are established: (1) If A and B are any two mutually exclusive, non-degenerate subsets of C, there exist two regions (i.e., connected open sets) R_1 and R_2 such that $\overline{R}_1 \cdot \overline{R}_2 = 0$, $A \cdot R_1 \neq 0 \neq B \cdot R_1$, and $A \cdot R_2 \neq 0 \neq B \cdot R_2$; (2) if C has no local separating point, then for each pair of points a, b of C there exists an infinite sequence of mutually exclusive regions $R_1, R_2, \cdots$, such that, for each i, $R_i + a + b$ is connected and locally connected and, for each i and j, $\overline{R}_i \cdot \overline{R}_j = a+b$; (3) there exists a space C which is both an absolute G_δ and an absolute F_δ and yet which is not cyclicly connected; (4) if C is an absolute G_δ and has no local separating point, each pair a, b of points of C can be joined in C by a continuum which is the sum of c independent arcs from a to b. (Received December 24, 1930.)

101. Professor G. T. Whyburn: *A c-point connected continuous curve containing no uncountable collection of mutually exclusive continua.*

A continuum is said to be c-point connected provided it is not disconnected by the omission of any countable subset. In this paper there is constructed a continuous curve (locally connected continuum) E in the plane which is c-point connected but which contains no uncountable collection of mutually exclusive continua. This example yields a second negative solution to a problem of Urysohn. Also the continuum E contains a G_δ set dense in E no point of which lies on any continuum of convergence in E and no two points of which can be joined by an infinity of independent continua. For a statement of the problem of Urysohn together with its solution, see Mazurkiewicz, Fundamenta Mathematicae, vol. 14, p. 222. The example of Mazurkiewicz, although possessing other striking properties, not only fails to be c-point connected but indeed it contains points at which it is rational, i.e., points which are contained in arbitrarily small neighborhoods having countable boundaries. (Received December 24, 1930.)

102. Dr. H. E. Stelson: *On double function space.*

A point of function space is determined by a single continuous infinity of coordinates, i.e., the values of $x(t)$ as t ranges over the interval (0, 1). By analogy a space in which each point is determined by n functions bears a relationship to a euclidean space of n dimensions. This investigation is concerned principally with the development of analytic geometry of a function space of two independent functions. The properties of the line and conic in this space are considered. These generalizations of euclidean space contain singularities not found in plane analytic geometry. Arbitrary functions entering these forms are evaluated by the aid of the calculus of variations. The generalized Schwarzian form $(fBfg)^2 = fAf^2fCg^2$ is discussed. A generalization of the above conic in accordance with the Fredholm transformation is discussed and it is pointed out that the theory of A. D. Michal applies to this function space. (Received December 24, 1930.)

103. Professor W. A. Hurwitz: *Oscillation of sequences and functions.*

This paper extends the author's former investigation (American Journal of Mathematics, vol. 62, p. 611) to the case of sequence to function transformations, not necessarily regular. The results supplement Knopp's work on the limit core of a sequence. (Received December 30, 1930.)

104. Professor W. B. Ford: *Two theorems on the partitions of numbers.*

Let $P^{(m)}(n)$ represent the number of ways in which the positive integer n may be expressed (repetitions being allowed) as the sum of mth powers of integers ($m=1, 2, 3, \cdots$). This paper shows the existence and form of a linear relation connecting the quantities $P^{(m)}(n), P^{(m)}(n-1), P^{(m)}(n-2), \cdots, P^{(m)}(1)$. Similarly, let $p^{(m)}(n)$ represents the number of ways in which n may be expressed when repetitions are *not* allowed as the sum of mth powers. This paper indicates the existence and form of a linear relation connecting the quantities $p^{(m)}(n), p^{(m)}(n-1), p^{(m)}(n-2), \cdots, p^{(m)}(1)$. Thus it appears that both $P^{(m)}(n)$ and $p^{(m)}(n)$ satisfy linear recurrence relations by means of which their successive values corresponding to increasing values of n may be rapidly determined from the values of $P^{(m)}(1)$ and $p^{(m)}(1)$. (Received December 30, 1930.)

105. Dr. R. P. Agnew (National Research Fellow): *On convergence and summability of double orthogonal series.*

Let A be a set of d-dimensional measure $m(A)>0$. Let $\{\phi_{mn}\}$ be a double sequence of real measurable functions, defined, normal, and orthogonal over A, and let $\{a_{mn}\}$ be a double sequence of real constants. The principal result of this paper is the following: If the coefficients in the double orthogonal series (1) $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}\phi_{ij}$ satisfy the condition (2) $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}[\log(i+1)\}]^2[\log(j+1)]^2 \cdot a_{ij}^2<\infty$, then (1) converges essentially uniformly over A. It is also shown that (2) ensures the existence of a subset of A of measure $m(A)$ over which each row and column of (1) converges and over which the series of values of the rows (or columns) converges to the value of (1). Some related theorems are given; one of these pertains to summability of orthogonal series. (Received December 22, 1930.)

106. Dr. A. B. Brown: *On the join of two complexes.*

In this paper the terms used are as defined in Lefschetz's *Topology* (vol. 12, Colloquium Series). Cells and spheres are combinatorial, hence not necessarily homeomorphic to ordinary cells and spheres. Lefschetz proves that the join of two closed cells, or of a closed cell and a sphere, is a closed cell. We prove that if the given cells or cell, as the case may be, are normal, then the resulting cell is likewise normal, that is, the join of a point and a sphere. We also obtain formulas for the Betti numbers of the join of any two complexes, in terms of those of the given complexes. The formulas hold for Betti numbers absolute or mod p, p any prime greater than unity. (Received January 6, 1931.)

107. Dr. N. H. McCoy (National Research Fellow): *On the resultant of three double binary forms.*

By using certain covariants which vanish for the common solutions of the forms equated to zero, T. W. Moore (Annals of Mathematics, (2), vol. 30 (1928), pp. 92–100) has expressed the resultant of three double binary forms as a determinant for a limited range of cases. Some of its properties are here obtained by an adaptation of Poisson's method, and the resultant is then expressed in the general case as the quotient of a determinant by a power of a single coefficient of one of the forms. In the case where one of the sets of variables enters each form to the same degree, the resultant is obtained as a determinant without the extraneous factor. (Received January 6, 1931.)

108. Professor Oystein Ore: *Linear equations in non-commutative domains.*

The problem of solving linear equations with coefficients from a non-commutative field has recently been studied by various authors, and analogies to the determinants have been introduced. The usefulness of these expressions is however inconveniently limited by the fact that certain restrictions must be placed on the elements involved. In this paper a new definition of non-commutative determinants is introduced; these determinants are general, have symmetric properties, and the solution of linear equations can be expressed in the same way as by commutative fields. The construction of quotient-fields of non-commutative rings is also discussed. (Received January 5, 1931.)

109. Dr. G. T. Whyburn: *On the cyclic connectivity theorem.*

The point p of the connected and locally connected separable metric space M is called a local end point of M provided p is a point of potential order 1 of some region (i.e., connected open set) in M. In this paper some properties of local end points are developed which are analogous to properties of the end points of continua. It is shown that if M has no cut point, then for any two non-local end points a and b, there exist two mutually exclusive regions R_1 and R_2 such that R_1+a+b and R_2+a+b are connected and locally connected and $\overline{R}_1\cdot\overline{R}_2=a+b$. The proof of this result suggests a simple proof for the theorem (called the cyclic connectivity theorem) that any two points of a locally compact space M without cut points lie together on a simple closed curve in M. This proof is based on a small amount of the cyclic element theory. (Received January 6, 1931.)

110. Professor W. A. Wilson: *On semi-metric spaces.*

This paper is concerned with spaces in which to each pair of points a and b there corresponds a positive distance $ab=ba$. Certain axioms weaker than the triangle axiom are discussed and in particular it is shown that a semi-metric space in which to each point a and each constant $k>0$ there corresponds an $r>0$ such that the relation $ab\geqq k$ implies $ac+bc\geqq r$ for every point c is homeomorphic with a metric space. The relations between semi-metric and topological spaces are discussed and new conditions obtained under which topological spaces are homeomorphic with metric spaces. (Received January 6, 1931.)

111. Dr. Jakob Levitzki: *On nilpotent subrings.*

In a recent paper (Mathematische Annalen, vol. 103) Koethe makes the conjecture that every nilring which lies in a ring of matrices (of finite degree) with elements from any field, must be nilpotent. In this paper, a proof for this conjecture is obtained, as a special case of a more general theorem concerning the structure of nilrings which lie in a ring in which the double-series postulate on right (or left) ideals holds. In an appendix to this paper the author gives a simple proof for a theorem of Koethe's concerning nilpotent subrings. (Received January 9, 1931.)

112. Dr. Jakob Levitzki: *A Galois theory in semi-simple rings.*

In this paper a semi-simple ring P and semi-simple subrings $\overline{P}$ of P over the same (commutative) field of automorphisms K are considered. The author defines the group Γ of all the automorphisms of P under which every element of K is invariant. He studies the correspondence between the subrings $\overline{P}$ of P and the subgroups Γ of $\overline{\Gamma}$. A characterization is obtained for the semi-simple subrings $\overline{P}$ for which the Galois theory holds. Similar results are also obtained for the subgroups of Γ. (Received January 9, 1931.)

113. Professor O. D. Kellogg: *On the derivatives of harmonic functions on the boundary.*

Numerous studies have been made of the behavior of the derivatives of the potentials of spreads of attracting matter on the surfaces bearing the spreads But comparatively little has been done on the behavior of the derivatives of harmonic functions defined by their boundary values, particularly in space of three dimensions (for the plane, see a paper by the author, Transactions of this Society, vol. 13 (1912), pp. 109–132). The main results of the present paper may be briefly characterized as follows. A Dini condition on the partial derivatives of order n of the boundary values of the harmonic function U, and on the corresponding derivatives of the functions defining the boundary, guarantees the existence, as limits from within, of the boundary values of the derivatives of order n of U. Thus defined, the derivatives are continuous on the boundary. Also, Hölder conditions on the boundary values of U, or on their derivatives of order n, are propagated into the interior of the region of definition of U. The results are local in character, in that hypotheses need be made only on a portion of the boundary. This portion, however, must have bounded curvatures. The characteristic tool is Poisson's integral. (Received January 5, 1931.)