ON THE EXTENSION OF THE GAUSS MEAN-VALUE THEOREM TO CIRCLES IN THE NEIGHBORHOOD OF ISOLATED SINGULAR POINTS OF HARMONIC FUNCTIONS

BY G. E. RAYNOR

1. *Introduction.* Let $f(x, y)$ be a function harmonic in a plane region R except at an isolated singular point P in R, and let C_1 be a circle in R with radius r_1 and with P as center. In previous papers* the writer has shown that in this neighborhood $f(x, y)$ can be put in the form

$$(1) \qquad f(x, y) = c \log \frac{1}{r} + \Phi(x, y) + V(x, y),$$

where†

$$c = \frac{1}{2\pi} \int_{C_1} \frac{\partial f}{\partial n} ds,$$

r being the distance from (x, y) to P, $\Phi(x, y)$, unless it be identically zero, harmonic in the neighborhood of P and such that there exist modes of approach to P for which the sum $c \log (1/r) + \Phi$ tends toward plus infinity and also toward minus infinity; and V is harmonic everywhere in the neighborhood of P including P. Also on C_1, $\Phi \equiv 0$. It is to be noticed that the constant c may be zero so that Φ has the same properties ascribed to the sum $c \log (1/r) + \Phi$.

If a system of polar coordinates (r, θ) be chosen with P as pole, Φ may be expanded, for $r \leqq r_1$, in the form‡

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* G. E. Raynor, *Isolated singular points of harmonic functions*, this Bulletin, vol. 32 (1926), p. 543, and *Integro-differential equations of the Bôcher type*, this Bulletin, vol. 32, p. 654.

† Here, as in all that follows, the normal derivatives are to be taken in the direction of the inner normal.

‡ G. E. Raynor, *Note on the expansion of harmonic functions in the neighborhood of isolated singular points*, Annals of Mathematics, vol. 31 (1930), p. 40. We shall refer to this as paper (A).

$$(2)\quad \Phi = \sum_{m=1}^{\infty} \frac{1}{m}\left[\left(\frac{r_1}{r}\right)^m - \left(\frac{r}{r_1}\right)^m\right](\gamma_m \cos m\theta + \delta_m \sin m\,\theta)$$

where

$$(3)\ \gamma_m = \frac{r_1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial \Phi_{r_1}}{\partial n} \cos m\theta d\theta;\quad \delta_m = \frac{r_1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial \Phi_{r_1}}{\partial n} \sin m\theta d\theta.$$

Also the two series

$$(4)\qquad G\left(\frac{r}{r_1},\ \theta\right) = -\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{r}{r_1}\right)^m (\gamma_m \cos m\theta + \delta_m \sin m\theta)$$

and

$$(5)\qquad G\left(\frac{r_1}{r},\ \theta\right) = \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{r_1}{r}\right)^m (\gamma_m \cos m\theta + \delta_m \sin m\theta)$$

are convergent for all values of θ and of $r \leqq r_1$,* and Φ can be expressed in the form

$$(6)\qquad \Phi = G\left(\frac{r}{r_1},\ \theta\right) + G\left(\frac{r_1}{r},\ \theta\right).$$

Furthermore

$$G\left(\frac{r}{r_1},\ \theta\right) = \frac{r_1}{4\pi}\int_{-\pi}^{\pi} \frac{\partial \Phi_{r_1}}{\partial n} \log\left[1 - 2\frac{r}{r_1}\cos(\alpha - \theta) + \frac{r^2}{r_1^2}\right] d\alpha,$$

which gives a solution of the Neumann problem for the circle C_1 with boundary values of the normal derivative equal to one-half the value of the normal derivative of Φ on C_1.† The function Φ also possesses the property

$$(7)\qquad \int_C \Phi ds = 0,$$

where C is any circle concentric with C_1 and of radius $r \leqq r_1$. In view of this last property it becomes of interest to inquire as to the value of

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* Paper (A), p. 40. Note that the definition of $G\{r/r_1\}$ of (21) of paper (A) has been slightly changed by inserting a minus sign in the right side of (4).

† Paper (A), p. 41; and Goursat, *Cours d'Analyse Mathématique*, vol. 3, 3d ed., p. 240.

$$\int_{C_2} \Phi ds$$

if C_2 lies within C_1 but does not have its center at P. Of course, if Q be the center of C_2 and P lies without C_2 we have by the Gauss mean-value theorem*

$$\frac{1}{2\pi r_2}\int_{C_2} \Phi ds = \Phi(Q),$$

where r_2 is the radius of C_2. Our purpose then, in this note, is to find the value of

$$\frac{1}{2\pi r_2}\int_{C_2} \Phi ds,$$

in the case of P within C_2. In §3 we shall also examine the mean value of $f(x, y)$ over C_2.

2. *The Mean Value of* Φ. Let a be the distance of Q from P and choose the line PQ as polar axis. Then by (2)

$$\Phi(a, \theta) = \sum_{m=1}^{\infty} \frac{1}{m}\left[\left(\frac{r_1}{a}\right)^m - \left(\frac{a}{r_1}\right)^m\right]\gamma_m \tag{8}$$

and hence by (4)

$$G\left(\frac{a}{r_1}, \theta\right) = -\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{a}{r_1}\right)^m \gamma_m. \tag{9}$$

By Green's formula, we have for the region bounded by the circles C_1 and C_2,

$$\begin{aligned}&\int_{C_1}\left(\Phi\frac{\partial \log r}{\partial n} - \log r\frac{\partial \Phi}{\partial n}\right)ds \\ &+ \int_{C_2}\left(\Phi\frac{\partial \log r}{\partial n} - \log r\frac{\partial \Phi}{\partial n}\right)ds = 0,\end{aligned} \tag{10}$$

where r is the distance from a variable point (x, y) on C_1 or C_2 to the center Q of C, and the normal derivatives are taken

* For a statement of the Gauss mean value theorem see Goursat, loc. cit., p. 181.

toward the interior of our region. Now on C_1, $\Phi \equiv 0$ and on C_2, $\partial \log r/\partial n = 1/r_2$. Furthermore

$$\int_{C_2} \frac{\partial \Phi}{\partial n} ds = 0.*$$

Hence, since $\log r$ is constant on C_2, relation (10) above reduces to

$$(11) \qquad \frac{1}{r_2}\int_{C_2} \Phi ds = \int_{C_1} \log r \frac{\partial \Phi}{\partial n} ds,$$

or

$$(12) \qquad \frac{1}{2\pi r_2}\int_{C_2} \Phi ds = \frac{r_1}{2\pi}\int_{-\pi}^{\pi} \log r \frac{\partial \Phi_{r_1}}{\partial n} d\theta.$$

Now from (1) we have

$$(13) \qquad \frac{\partial f_{r_1}}{\partial n} = \frac{c}{r_1} + \frac{\partial \Phi_{r_1}}{\partial n} + \frac{\partial V_{r_1}}{\partial n}.$$

Since f is harmonic *on* C_1 the left side of (13) has a derivative with respect to θ. Now V may be written as a Poisson integral and this integral may be expressed as the sum of a constant and the potential of a double layer. Since the values of V on C_1 are the values of f on C_1 diminished by the constant $c \log (1/r_1)$ the density of this double layer is analytic on C_1 and hence V is analytic in the *closed* region bounded by C_1 and therefore the third term on the right of (13) has a derivative with respect to θ.† Since the same is true, obviously, of the first term it follows that the second term $\partial\Phi_{r_1}/\partial n$ is also differentiable with respect to θ and hence is of bounded variation. $\partial\Phi_{r_1}/\partial n$ thus satisfies the conditions for expansion in a Fourier series and furthermore this series will be uniformly convergent in the closed interval $-\pi$ to π. We thus have

$$(14) \qquad \frac{\partial \Phi_{r_1}}{\partial n} = \frac{2}{r_1}\sum_{m=1}^{\infty}(\gamma_m \cos m\theta + \delta_m \sin m\theta),$$

where γ_m and δ_m are as given in (3). The constant term in the expansion of $\partial\Phi_{r_1}/\partial n$ is zero since as stated previously

* See the second paper of the first footnote.

† Encyklopädie der Mathematischen Wissenschaften, vol. 2, 3, 1, p. 206.

$$\int_{C_1} \frac{\partial \Phi_{r_1}}{\partial n} ds = 0.$$

It may be pointed out that (14) may be obtained by differentiating (2) in the direction of the inner normal and then allowing r to approach r_1.

Since $\log r$ is bounded on C_1 it follows from (14) that the series

$$(15)\quad \log r \frac{\partial \Phi_{r_1}}{\partial n} = \frac{2}{r_1} \sum_{m=1}^{\infty} (\gamma_m \log r \cos m\theta + \delta_n \log r \sin m\theta)$$

is uniformly convergent on C_1. Hence the series may be integrated termwise and we have

$$(16)\quad \frac{r_1}{2\pi} \int_{-\pi}^{\pi} \log r \frac{\partial \Phi_{r_1}}{\partial n} d\theta = \frac{1}{\pi} \sum_{m=1}^{\infty} \left[\gamma_m \int_{-\pi}^{\pi} \log r \cos m\theta d\theta + \delta_m \int_{-\pi}^{\pi} \log r \sin m\theta d\theta \right].$$

But

$$r^2 = r_1^2 - 2r_1 a \cos\theta + a^2,$$

and hence

$$(17)\qquad \log r = \log r_1 + \frac{1}{2} \log \left(1 - 2\frac{a}{r_1} \cos\theta + \frac{a^2}{r_1}\right).$$

Now*

$$(18)\quad \int_{-\pi}^{\pi} \log \left(1 - 2\frac{a}{r_1} \cos\theta + \frac{a^2}{r_1^2}\right) \cos m\theta d\theta = -\frac{2\pi}{m} \left(\frac{a}{r_1}\right)^m.$$

Also

$$(19)\qquad \int_{-\pi}^{\pi} \log \left(1 - 2\frac{a}{r_1} \cos\theta + \frac{a^2}{r_1^2}\right) \sin m\theta d\theta = 0.$$

Hence by (17), (18), and (19), equation (16), since $\log r_1$ is a constant, takes the form

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* Edwards, *The Integral Calculus*, vol. 2, p. 306, formula (10). The integrand in equation (18) above takes the same value in the interval $-\pi$ to 0 as in the interval 0 to π and hence Edwards' result must be multiplied by 2.

$$(20)\qquad \frac{r_1}{2\pi}\int_{-\pi}^{\pi}\log r\frac{\partial\Phi_{r_1}}{\partial n}d\theta = -\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{a}{r_1}\right)^m\gamma_m = G\left(\frac{a}{r_1},\ \theta\right).$$

Thus equation (12) combined with (20) gives

$$\frac{1}{2\pi r_2}\int_{C_2}\Phi ds = G\left(\frac{a}{r_1},\ \theta\right);$$

and we can state the following theorem.

THEOREM 1. *The mean value of the function Φ of equation* (1) *over a circle C_2 within C_1 having the singular point P in its interior is equal to the value of the function $G\{(r/r_1),\ \theta\}$ at Q, where Q is the center of C_2.*

Since $G\{(r/r_1),\ \theta\}$ is harmonic everywhere within C_2 its value at Q by Gauss's theorem is its mean value over C_2 and hence Theorem 1 can be stated in the form:

THEOREM 2. *The mean value of the function Φ over a circle C_2 within C_1, having the singular point P in its interior is equal to the mean value of the function $G\{(r/r_1),\ \theta\}$ over C_2.*

Since

$$\Phi(r,\ \theta) = G\left(\frac{r}{r_1},\ \theta\right) + G\left(\frac{r_1}{r},\ \theta\right),$$

it follows from Theorem 2 that we have the result:

THEOREM 3. *The mean value of the function $G\{(r_1/r),\ \theta\}$ over a circle C_2 within C_1, having P in its interior, is zero.*

It is to be noticed that the above theorems are true if the center Q of C coincides with P.

3. *The Mean Value of $f(x, y)$.* If we wish to find the mean value of $f(x, y)$ over C_2 we must add to the mean value of Φ the mean values over C_2 of the first and third terms of equation (1). Now the first term $c \log (1/r)$ is readily seen from (17) to be equivalent to

$$(21)\qquad -c\log r_1 - \frac{c}{2}\log\left(1 - 2\frac{a}{r_1}\cos\theta + \frac{a^2}{r_1^2}\right).$$

But we have*

$$\int_{-\pi}^{\pi} \log\left(1 - 2\frac{a}{r_1}\cos\theta + \frac{a^2}{r_1^2}\right)d\theta = 0. \tag{22}$$

Thus the mean value of the first term of (1) over C_2 is $-c \log r_1$. Now the third term $V(x, y)$ is harmonic in C_2 and as stated previously is equal to

$$U(x, y) + c_1 \log r_1,$$

where $U(x, y)$ takes the same values on C_1 as $f(x, y)$. Hence the mean value of $V(x, y)$ over C_2 is equal to

$$U(Q) + c_1 \log r_1.$$

We thus find the mean value of the sum of the first and third terms of (1) to be $U(Q)$. Combining this result with the theorem of the previous section we have the theorem:

THEOREM 4. *The mean value of the function f of equation* (1) *over C_2 is equal to $U(Q)$ plus the value of the function*

$$G\left(\frac{r}{r_1}, \theta\right) = \frac{r_1}{4\pi}\int_{-\pi}^{\pi} \frac{\partial \Phi_{r_1}}{\partial n} \log\left[1 - 2\frac{r}{r_1}\cos(\alpha - \theta) + \frac{r^2}{r_1^2}\right]d\alpha$$

at Q, where U is the function which solves the Dirichlet problem† *for C_1 with boundary values f, and $G\{(r/r_1), \theta\}$ is a solution of the Neumann problem for C_1 with boundary values of the normal derivatives equal to one-half the normal derivatives of Φ on C_1.*

It is to be noticed that if P is not a singular point, the above theorem reduces to the Gauss mean-value theorem.

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* Edwards, loc. cit., p. 306, formula (9). The integrand in (22) above takes the same values in the interval $-\pi$ to 0 as in the interval 0 to π and hence Edwards' result gives zero for (22).

† For a statement of the Dirichlet problem see Goursat, loc. cit., p. 196.