# THEOREMS ON INVERTED AND ROTATED CONGRUENCES*

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1. *Introduction*. Let $l$ be any line of a rectilinear congruence and $M$ that point on the unit sphere $S$ at which the normal is parallel to $l$. We refer the sphere to any isothermal system and take the linear element in the form

$$(1) \qquad ds^2 = e^{2\lambda}(du^2 + dv^2).$$

At $M$ we consider the moving trihedral of $S$ whose $x$-axis is chosen tangent to the curves $v = \text{const.}$, and let $(a, b)$ be the coördinates of the point in which $l$ meets the $xy$-plane. The conditions that a congruence $l$ be of a particular type are conditions upon the functions $a$ and $b$.†

For the sphere, $F = D' = 0$, and hence

$$(2) \qquad \xi_1 = \eta = p = q_1 = 0.$$

Also $E = \mathcal{E}$, $G = \mathcal{G}$, and $\rho_1 = \rho_2 = -1$;‡ consequently $D = -E$, $D'' = -G = -E$. Hence we readily find§

$$(3) \qquad \xi = \eta_1 = q = -p_1 = e^{\lambda},\ r = -\frac{\partial\lambda}{\partial v},\quad r_1 = \frac{\partial\lambda}{\partial u}.$$

When the functions in (3) are substituted in the six fundamental relations which are the equivalent of the Gauss and Codazzi equations,|| all except the equation

$$(4) \qquad \frac{\partial^2\lambda}{\partial u^2} + \frac{\partial^2\lambda}{\partial v^2} = -e^{2\lambda}¶$$

* Presented to the Society, April 18, 1930.

† Malcolm Foster, *Rectilinear congruences referred to special surfaces*, Annals of Mathematics, (2), vol. 25 (1923), pp. 159–180.

‡ The positive direction of the normal is chosen outward.

§ Eisenhart, *Differential Geometry of Curves and Surfaces*, p. 174.

|| Eisenhart, p. 168 and p. 170.

¶ Foster, loc. cit. See p. 160 for the solution of this equation.

are identically satisfied. Any solution of (4) determines a particular isothermal system on the sphere $S$ and two adjoint minimal surfaces $S_1$ and $S_2$, such that the isothermal system on $S$ is the representation of the lines of curvature on $S_1$ and of the asymptotic lines on $S_2$.*

We shall also associate each line of a congruence with those points on the surfaces $S_1$ and $S_2$ at which the normals have the same direction. When $S_1$ and $S_2$ are referred to their lines of curvature and asymptotic lines respectively, and the linear element of each is taken in the form

$$ds^2 = e^{-2\lambda}(du^2 + dv^2), \tag{5}$$

we readily find that the fundamental quantities for $S_1$ and $S_2$ are as follows.†

| For $S_1$ | For $S_2$ |
|---|---|
| $\xi_1 = \eta = p = q_1 = 0,$ | $\xi_1 = \eta = p_1 = q = 0,$ |
| (6) $\xi = \eta_1 = e^{-\lambda},\quad p_1 = q = -e^{\lambda},$ | $\xi = \eta_1 = e^{-\lambda},\ p = -q_1 = e^{\lambda},$ |
| $r = \dfrac{\partial\lambda}{\partial v},\qquad r_1 = -\dfrac{\partial\lambda}{\partial u}\cdot$ | $r = \dfrac{\partial\lambda}{\partial v},\qquad r_1 = -\dfrac{\partial\lambda}{\partial u}\cdot$ |

2. *The Inversion of a Congruence.* Consider any line $l$ of a congruence and the moving trihedral at the associated point $M$ of $S$. Relative to a circle $C$ lying in the tangent plane with center at $M$ and of constant radius $k$, let us invert the point $(a, b)$ in which $l$ meets the tangent plane. The lines $l_1$ drawn through the inverted points

$$\left(a_1 = \frac{ka}{a^2 + b^2},\quad b_1 = \frac{kb}{a^2 + b^2}\right), \tag{7}$$

parallel to the normals at the associated points $M$ will constitute a congruence which we call the inverse of the congruence formed by the lines $l$.

Let us suppose that the congruence $l$ is isotropic. Then

$$a = e^{\lambda}\frac{\partial Q}{\partial u},\ b = -e^{\lambda}\frac{\partial Q}{\partial v},\quad \frac{\partial^2 Q}{\partial u^2} + \frac{\partial^2 Q}{\partial v^2} = 0.\ddagger \tag{8}$$

* Eisenhart, p. 252.

† Foster, loc. cit., p. 173.

‡ Foster, loc. cit., p. 173.

We ask: Under what conditions will the congruence $l_1$ be normal? The condition that a congruence referred to $S$ be normal is*

$$a = e^{-\lambda}\frac{\partial P}{\partial u}, \quad b = e^{-\lambda}\frac{\partial P}{\partial v}, \tag{9}$$

where $P(u, v)$ is an arbitrary function. Hence from (7), (8), and (9), the condition that the congruence $l_1$ be normal is

$$\frac{k\dfrac{\partial Q}{\partial u}}{\left(\dfrac{\partial Q}{\partial u}\right)^2 + \left(\dfrac{\partial Q}{\partial v}\right)^2} = \frac{\partial P}{\partial u}, \quad \frac{-k\dfrac{\partial Q}{\partial v}}{\left(\dfrac{\partial Q}{\partial u}\right)^2 + \left(\dfrac{\partial Q}{\partial v}\right)^2} = \frac{\partial P}{\partial v}, \tag{10}$$

where we must have

$$\frac{\partial^2 P}{\partial u \partial v} = \frac{\partial^2 P}{\partial v \partial u}. \tag{11}$$

It is readily found from (10), on making use of (8), that (11) is satisfied identically. Hence we have the following theorem.

THEOREM 1. *If an isotropic congruence $l$ referred to $S$ be inverted relative to $C$, the inverted congruence $l_1$ is normal.*

The condition that a congruence referred to $S$ have the center of the sphere for its middle envelope is†

$$a = e^{-\lambda}\frac{\partial R}{\partial v}, \quad b = -e^{-\lambda}\frac{\partial R}{\partial u}, \tag{12}$$

where $R(u, v)$ is an arbitrary function. Let us again invert an isotropic congruence and ask when the inverted congruence $l_1$ will be of the type (12). From (7), (8), and (12) this condition will be

$$\frac{k\dfrac{\partial Q}{\partial u}}{\left(\dfrac{\partial Q}{\partial u}\right)^2 + \left(\dfrac{\partial Q}{\partial v}\right)^2} = \frac{\partial R}{\partial v}, \quad \frac{k\dfrac{\partial Q}{\partial v}}{\left(\dfrac{\partial Q}{\partial u}\right)^2 + \left(\dfrac{\partial Q}{\partial v}\right)^2} = \frac{\partial R}{\partial u}, \tag{13}$$

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* Foster, loc. cit., p. 173.

† Foster, loc. cit., p. 173.

where we must have $\partial^2 R/\partial u\partial v = \partial^2 R/\partial u\partial v$. As before we readily find on using (8) that this latter relation is satisfied identically. Hence we have the following theorem.

THEOREM 2. *If an isotropic congruence $l$ referred to $S$ be inverted relative to $C$, the inverted congruence $l_1$ is normal, and has for its middle envelope the center of $S$.*

The surfaces normal to $l_1$ have been considered by Appell.*

3. *The Rotation of Congruences.* We now refer the congruence $l$ to the minimal surface $S_1$ whose lines of curvature are parametric. The equation whose roots $t_1$ and $t_2$ give the distances of the focal points from the tangent plane is†

$$q^2t^2 + q\left(\frac{\partial a}{\partial u} - \frac{\partial b}{\partial v} - ar_1 - br\right)t + \left(\frac{\partial a}{\partial v} - br_1\right)\left(\frac{\partial b}{\partial u} + ar\right)$$
$$- \left(\frac{\partial a}{\partial u} + \xi - br\right)\left(\frac{\partial b}{\partial v} + \xi + ar_1\right) = 0.$$

Hence the necessary and sufficient condition that $S_1$ be the middle envelope of the congruence $l$ is that $t_1+t_2=0$, or

$$\frac{\partial a}{\partial u} - \frac{\partial b}{\partial v} - ar_1 - br = 0. \tag{14}$$

On using (6) and multiplying by $e^\lambda$ this relation becomes

$$\frac{\partial}{\partial u}(ae^\lambda) = \frac{\partial}{\partial v}(be^\lambda). \tag{15}$$

Let us set each member of (15) equal to $\partial^2 L/\partial u\partial v$, where $L$ is an arbitrary function of $u$ and $v$. Then the necessary and sufficient condition that $S_1$ be the middle envelope of $l$ is

$$a = e^{-\lambda}\frac{\partial L}{\partial v}, \qquad b = e^{-\lambda}\frac{\partial L}{\partial u}. \tag{16}$$

On comparing this with the condition that a congruence referred to $S_1$ be normal,‡

* P. Appell, American Journal of Mathematics, vol. 10 (1888), p. 175.

† Foster, loc. cit., p. 170.

‡ Foster, loc. cit., p. 173.

$$a = e^{-\lambda}\frac{\partial P}{\partial u}, \quad b = -e^{-\lambda}\frac{\partial P}{\partial v}, \tag{17}$$

where $P(u, v)$ is an arbitrary function, we have the following theorem.

THEOREM 3. *If the lines of a normal congruence referred to $S_1$ be rotated through an angle $\pi/2$ about the corresponding normals, the middle envelope of the resulting congruence will be $S_1$.**

If a congruence is referred to the minimal surface $S_2$ with its asymptotic lines parametric, the equation for the distances from the tangent plane to the focal points is†

$$p^2t^2 - p\left(\frac{\partial a}{\partial v} + \frac{\partial b}{\partial u} + ar - br_1\right)t + \left(\frac{\partial a}{\partial v} - br_1\right)\left(\frac{\partial b}{\partial u} + ar\right)$$
$$- \left(\frac{\partial a}{\partial u} + \xi - br\right)\left(\frac{\partial b}{\partial v} + \xi + ar_1\right) = 0.$$

The necessary and sufficient condition that $S_2$ be the middle envelope is

$$\frac{\partial a}{\partial v} + \frac{\partial b}{\partial u} + ar - br_1 = 0,$$

which becomes on using (6) and multiplying by $e^{\lambda}$,

$$\frac{\partial}{\partial v}(ae^{\lambda}) = -\frac{\partial}{\partial u}(be^{\lambda}). \tag{18}$$

On setting each member of (18) equal to $\partial^2 M/\partial u \partial v$, where $M$ is an arbitrary function of $u$ and $v$, we find that the necessary and sufficient condition that $S_2$ be the middle envelope of the given congruence is

$$a = e^{-\lambda}\frac{\partial M}{\partial u}, \quad b = -e^{-\lambda}\frac{\partial M}{\partial v}. \tag{19}$$

The condition that a congruence referred to $S_2$ be normal is‡

* The direction of rotation is evidently immaterial, since $P$ in (17) is arbitrary in sign.

† Foster, loc. cit., p. 172.

‡ Foster, loc. cit., p. 173.

$$a = e^{-\lambda}\frac{\partial P}{\partial v}, \quad b = e^{-\lambda}\frac{\partial P}{\partial u}, \tag{20}$$

where the function $P(u, v)$ is arbitrary; hence on comparing (19) and (20), we have the following theorem.

THEOREM 4. *If the lines of a normal congruence referred to* $S_2$ *be rotated through an angle* $\pi/2$ *about the corresponding normals, the middle envelope of the resulting congruence will be* $S_2$.

Let us now seek the conditions under which the normal congruence (17), referred to $S_1$, will remain normal after a rotation through an angle $\pi/2$. From (17) we must obviously have

$$\frac{\partial P}{\partial v} = \frac{\partial K}{\partial u}, \quad \frac{\partial P}{\partial u} = -\frac{\partial K}{\partial v},$$

where $K(u, v)$ is arbitrary. Hence from these equations the condition is that $P$ satisfy Laplace's equation $\partial^2 P/\partial u^2 + \partial^2 P/\partial v^2 = 0$. We obtain the same result when we make a similar inquiry concerning the rotation of the normal congruence (20) referred to $S_2$. Consequently we have the theorem:

THEOREM 5. *If the lines of a normal congruence referred to* $S_1$, $(S_2)$, *be rotated through an angle of* $\pi/2$ *about the corresponding normals, the resulting congruence will also be normal if the function* $P$ *in* (17), [(19)], *is a solution of Laplace's equation, and will have* $S_1$, $(S_2)$, *for its middle envelope.*

Since (16) is identical with (20), and (17) identical with (19), we have at once the following theorem.

THEOREM 6. *The point* $(a, b)$ *defining a normal congruence referred to* $S_1$, $(S_2)$, *when plotted with reference to the trihedral on* $S_2$, $(S_1)$, *defines a congruence whose middle envelope is* $S_2$, $(S_1)$.

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