GEODESIC COORDINATES OF ORDER r*

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1. *Introduction.* Let $\Gamma^i_{\alpha\beta}$ be a general symmetric affine connection† and $\Gamma^i_{\beta_1\beta_2\cdots\beta_r}$, $(r=3,\cdots,p)$, the sequence of the first $p-2$ generalized symmetric affine connections defined by‡

$$\Gamma^i_{\alpha\beta\gamma} = \frac{1}{3}P\left(\frac{\partial\Gamma^i_{\alpha\beta}}{\partial x^\gamma} - 2\Gamma^i_{\sigma\alpha}\Gamma^\sigma_{\beta\gamma}\right)$$

and in general by the recurrence formula

$$\Gamma^i_{\alpha\beta\gamma\cdots\delta\epsilon} = \frac{1}{N}P\left(\frac{\partial\Gamma^i_{\alpha\beta\gamma\cdots\delta}}{\partial x^\epsilon} - \Gamma^i_{\sigma\beta\gamma\cdots\delta}\Gamma^\sigma_{\alpha\epsilon} - \cdots - \Gamma^i_{\alpha\beta\gamma\cdots\sigma}\Gamma^\sigma_{\delta\epsilon}\right).$$

The law of transformation of the affine connection $\Gamma^i_{\alpha\beta}$ is, as is well known,

$$\overline{\Gamma}^\sigma_{\alpha\beta}\frac{\partial x^i}{\partial\bar{x}^\sigma} = \Gamma^i_{\lambda\mu}\frac{\partial x^\lambda}{\partial\bar{x}^\alpha}\frac{\partial x^\mu}{\partial\bar{x}^\beta} + \frac{\partial^2 x^i}{\partial\bar{x}^\alpha\partial\bar{x}^\beta},$$

while the generalized affine connections transform in accordance with the law§

$$\overline{\Gamma}^\sigma_{\alpha\cdots\beta}\frac{\partial x^i}{\partial\bar{x}^\sigma} = \frac{\partial^p x^i}{\partial\bar{x}^\alpha\cdots\partial\bar{x}^\beta} + \Gamma^i_{\lambda\cdots\mu}\frac{\partial x^\lambda}{\partial\bar{x}^\alpha}\cdots\frac{\partial x^\mu}{\partial\bar{x}^\beta} + [\quad]$$

where the [ ] denotes the sum of terms, each of which involves a component $\overline{\Gamma}^i_{\rho\cdots\sigma}$ with less than p subscripts, that vanish with $\overline{\Gamma}^i_{\rho\alpha},\cdots,\overline{\Gamma}^i_{\rho\cdots\sigma\beta}$.

2. *Fundamental Theorems.*

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* Presented to the Society, December 30, 1929.

† We assume that the reader is conversant with the tensor theory as presented by O. Veblen in his *Invariants of Quadratic Differential Forms*, Cambridge Tract, 1927.

‡ O. Veblen and T. Y. Thomas, Transactions of this Society, vol. 25 (1923), p. 561.

§ T. Y. Thomas, American Journal Mathematics, vol. 50 (1928), p. 518.

THEOREM 1. *A necessary and sufficient condition that a coordinate system y^i determined by a coordinate system x^i and a point $x^i=q^i$ as origin have the property that $\Gamma^i_{\alpha\beta}$, $\Gamma^i_{\alpha\beta\gamma}, \cdots,$ $\Gamma^i_{\alpha\beta\gamma\cdots\kappa}$ all vanish at the origin $y^i=0$ when evaluated in the y^i system of coordinates is that*

$$(1)\left\{\begin{array}{l} \dfrac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta} = -\Gamma^i_{\lambda\mu}(x)\dfrac{\partial x^\lambda}{\partial y^\alpha}\dfrac{\partial x^\mu}{\partial y^\beta}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{\partial^p x^i}{\partial y^\alpha \partial y^\beta \partial y^\gamma \cdots \partial y^\delta} = -\Gamma^i_{\lambda\mu\nu\cdots\rho}(x)\dfrac{\partial x^\lambda}{\partial y^\alpha}\dfrac{\partial x^\mu}{\partial y^\beta}\dfrac{\partial x^\nu}{\partial y^\gamma}\cdots\dfrac{\partial x^\rho}{\partial y^\delta}, \end{array}\right.$$

for $x^i=q^i$

If a star over a function denotes the evaluation of that function in the preferred system y^i, we obviously have

$$(2)\left\{\begin{array}{l} {}^*\Gamma^\sigma_{\alpha\beta}(y)\dfrac{\partial x^i}{\partial y^\sigma} = \Gamma^i_{\lambda\mu}(x)\dfrac{\partial x^\lambda}{\partial y^\alpha}\dfrac{\partial x^\mu}{\partial y^\beta} + \dfrac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ {}^*\Gamma^\sigma_{\alpha\cdots\delta}\dfrac{\partial x^i}{\partial y^\sigma} = \Gamma^i_{\lambda\cdots\rho}\dfrac{\partial x^\lambda}{\partial y^\alpha}\cdots\dfrac{\partial x^\rho}{\partial y^\delta} + \dfrac{\partial^p x^i}{\partial y^\alpha \cdots \partial y^\delta} + [\ \ ] \end{array}\right.$$

where the [ ] on the right hand side of (2) denotes the sum of terms, each of which involves a component ${}^*\Gamma^i_{\rho\ldots\sigma}$ with less than p subscripts, that vanish with ${}^*\Gamma^i_{\rho\alpha}, \cdots, {}^*\Gamma^i_{\rho\ldots\beta}$. It is obvious now from (2) that a necessary and sufficient condition that

$$({}^*\Gamma^i_{\alpha\beta})_0 = 0,\ ({}^*\Gamma^i_{\alpha\beta\gamma})_0 = 0, \cdots,\ ({}^*\Gamma^{i'}_{\alpha\beta\cdots\delta})_0 = 0$$

is that conditions (1) hold at the *origin* $x^i=q^i$.

DEFINITION. *A coordinate system y^i for which*

$$({}^*\Gamma^i_{\alpha\beta})_0 = 0,\ ({}^*\Gamma^i_{\alpha\beta_1\beta_2})_0 = 0, \cdots,\ ({}^*\Gamma^i_{\alpha\beta_1\beta_2\cdots\beta_r})_0 = 0$$

will be called a geodesic coordinate system of order r.

THEOREM 2. *A geodesic coordinate system y^i of order r for which*

$$\left(\frac{\partial x^i}{\partial y^j}\right)_0 = \delta_j^{\ i}$$

is defined implicitly by the coordinate transformation

$$(3)\left\{\begin{aligned} x^i = q^i + y^i - \frac{1}{2!}(\Gamma^i_{\alpha\beta}(x))_q y^\alpha y^\beta - \frac{1}{3!}(\Gamma^i_{\alpha\beta\gamma}(x))_q y^\alpha y^\beta y^\gamma - \cdots \\ - \frac{1}{(r+1)!}(\Gamma^i_{\alpha\beta_1\cdots\beta_r}(x))_q y^\alpha y^{\beta_1} \cdots y^{\beta_r} + \psi^i(y) \end{aligned}\right.$$

where ψ^i is a regular power series in y^i which begins with powers of y^i at least as great as $r+2$.*

Successive differentiations of (3) and evaluations for $y^i=0$ together with an application of Theorem 1 establishes our theorem.

From now on we shall deal exclusively with geodesic coordinate systems of the particular type considered in Theorem 2.

THEOREM 3. *If*

$$\bar{x}^i = f^i(x^1, \cdots, x^n)$$

is an arbitrary analytic transformation of the coordinates x^i, then any two geodesic coordinate systems of order r, y^i and $\bar{y}^i$, with the same origin $x^i=q^i$, that are determined by (3) and

$$\bar{x}^i = \bar{q}^i + \bar{y}^i - \frac{1}{2!}(\overline{\Gamma}^i_{\alpha\beta}(\bar{x}))_{\bar{q}}\bar{y}^\alpha\bar{y}^\beta - \cdots$$
$$- \frac{1}{(r+1)!}(\overline{\Gamma}^i_{\alpha\beta_1\cdots\beta_r}(\bar{x}))_{\bar{q}}\bar{y}^\alpha\bar{y}^{\beta_1}\cdots\bar{y}^{\beta_r} + \pi^i(\bar{y})$$

respectively, are related by the transformation

$$(4)\qquad \bar{y}^i = \left(\frac{\partial \bar{x}^i}{\partial x^\alpha}\right)_q y^\alpha + \lambda^i(y)$$

where λ^i is a power series in y^α that begins with powers of y^α not less than $r+2$.

To prove this theorem we observe first that

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* We shall understand that $\psi^i \equiv 0$ is an admissible case.

$$\frac{\partial \bar{y}^i}{\partial y^\alpha} = \frac{\partial \bar{y}^i}{\partial \bar{x}^j}\frac{\partial \bar{x}^j}{\partial x^k}\frac{\partial x^k}{\partial y^\alpha}. \tag{5}$$

On solving the equations (3) for y^i in the neighborhood of $x^i = q^i$ we obtain

$$y^i = x^i - q^i + h^i$$

where the h^i are power series in $x^i - q^i$ beginning with the second degree terms in $x^i - q^i$. Similarly

$$\bar{y}^i = \bar{x}^i - \bar{q}^i + k^i \tag{6}$$

where k^i involves terms in $\bar{x}^i - \bar{q}^i$ of higher order than the first. Hence, evaluating (5) at the common origin of the preferred coordinate systems y^i and $\bar{y}^i$, we obtain

$$\left(\frac{\partial \bar{y}^i}{\partial y^\alpha}\right)_0 = \left(\frac{\partial \bar{x}^i}{\partial x^\alpha}\right)_q \tag{7}$$

on making use of the relations

$$\left(\frac{\partial x^i}{\partial y^\alpha}\right)_0 = \delta^i_\alpha, \qquad \left(\frac{\partial \bar{y}^i}{\partial \bar{x}^\alpha}\right) = \delta^i_\alpha.$$

By a similar method we also obtain the result

$$\left(\frac{\partial y^i}{\partial \bar{y}^\alpha}\right)_0 = \left(\frac{\partial x^i}{\partial \bar{x}^\alpha}\right)_{\bar{q}}.$$

The formulas that connect the components of the affine connection and the first $r-1$ generalized affine connections in the coordinate system y^i with those in the coordinate system $\bar{y}^i$ are

$$\tag{8}\left\{\begin{aligned} &{}^*\Gamma^\sigma_{\alpha\beta}\frac{\partial \bar{y}^i}{\partial y^\sigma} = {}^*\bar{\Gamma}^i_{\lambda\mu}\frac{\partial \bar{y}^\lambda}{\partial y^\alpha}\frac{\partial \bar{y}^\mu}{\partial y^\beta} + \frac{\partial^2 \bar{y}^i}{\partial y^\alpha \partial y^\beta},\\ &{}^*\Gamma^\sigma_{\alpha\cdots\delta}\frac{\partial \bar{y}^i}{\partial y^\sigma} = {}^*\bar{\Gamma}^i_{\lambda\cdots\mu}\frac{\partial \bar{y}^\lambda}{\partial y^\alpha}\cdots\frac{\partial \bar{y}^\mu}{\partial y^\delta} + \frac{\partial^p \bar{y}^i}{\partial y^\alpha\cdots\partial y^\delta}\\ &\qquad\qquad + [\quad](p = 3, 4, \cdots, r+1), \end{aligned}\right.$$

where the [ ] stands for a sum of terms that vanish with all the ${}^*\Gamma^\sigma_{\alpha\cdots\delta}$ in which the number of indices $\alpha, \cdots, \delta$ is less than p.

Evaluating (8) at the common origin of the coordinate systems y^i and $\bar{y}^i$ we obtain

$$
(9) \qquad \left(\frac{\partial^p \bar{y}^i}{\partial y^{\alpha_1} \cdots \partial y^{\alpha_p}}\right)_0 = 0, \qquad (p = 2, 3, \ldots, r+1).
$$

Our theorem then follows immediately from this result and the relations (7).

3. *Applications.* Let $T^{a \cdots b}_{c \cdots d}(x)$ be the components of a tensor and $*T^{a \cdots b}_{c \cdots d}(y)$ the components of the same tensor in a geodesic coordinate system y^i of order r with origin at $x^i = q^i$. A set of functions $T^{a \cdots b}_{c \cdots d, e \cdots f}$ of $x^1, \cdots, x^n$ at an arbitrarily given point $x^i = q^i$ will be defined by

$$
(T^{a \cdots b}_{c \cdots d, e_1 \cdots e_k}(x))_q = \left(\frac{\partial^k * T^{a \cdots b}_{c \cdots d}(y)}{\partial y^{e_1} \cdots \partial y^{e_k}}\right)_0 .
$$

THEOREM 4. *The functions $T^{a \cdots b}_{c \cdots d,\ e_1 \cdots e_k}(x^1, \cdots, x^n)$ for a fixed $k(\leqq r)$ are the components of a tensor.*

Let y^i and $\bar{y}^i$ be geodesic coordinates of order r with $x^i = q^i$ as their common origin. By hypothesis

$$
*\overline{T}^{\alpha \cdots \beta}_{\gamma \cdots \delta}(\bar{y}) = *T^{a \cdots b}_{c \cdots d}(y) \frac{\partial y^c}{\partial \bar{y}^\gamma} \cdots \frac{\partial y^d}{\partial \bar{y}^\delta} \frac{\partial \bar{y}^\alpha}{\partial y^a} \cdots \frac{\partial \bar{y}^\beta}{\partial y^b}.
$$

Hence by differentiation and Theorem 3, we have

$$
(10) \qquad \frac{\partial^k * \overline{T}^{\alpha \cdots \beta}_{\gamma \cdots \delta}(\bar{y})}{\partial \bar{y}^{\epsilon_1} \cdots \partial \bar{y}^{\epsilon_k}} = \frac{\partial^k * T^{a \cdots b}_{c \cdots d}(y)}{\partial y^{e_1} \cdots \partial y^{e_k}} \frac{\partial y^c}{\partial \bar{y}^\gamma} \cdots \frac{\partial y^d}{\partial \bar{y}^\delta} \frac{\partial y^{e_1}}{\partial \bar{y}^{\epsilon_1}} \cdots
$$

$$
\cdot \frac{\partial y^{e_k}}{\partial \bar{y}^{\epsilon_k}} \frac{\partial \bar{y}^\alpha}{\partial y^a} \cdots \frac{\partial \bar{y}^\beta}{\partial y^b} + [\quad]
$$

where [ ] denotes the sum of a set of terms that vanish at the common origin of the coordinate systems y^i and $\bar{y}^i$. By Theorem 3, we see that

$$
\left(\frac{\partial \bar{y}^i}{\partial y^j}\right)_0 = \left(\frac{\partial \bar{x}^i}{\partial x^j}\right)_q, \qquad \left(\frac{\partial y^i}{\partial \bar{y}^j}\right)_0 = \left(\frac{\partial x^i}{\partial \bar{x}^j}\right)_{\bar{q}}.
$$

If then we evaluate (10) at the origin and make use of the above relations in the resultant expression we obtain the law of transformation of a tensor evaluated at an arbitrarily given point. The proof of our theorem is therefore complete.

It is to be observed* that the tensor $T^{a\cdots b}_{c\cdots d,\ e}$ is the first affine extension (covariant derivative) of the tensor $T^{a\cdots b}_{c\cdots d}$ while $T^{a\cdots b}_{c\cdots d,\ e_1\cdots e_k}$ is the kth affine extension of $T^{a\cdots b}_{c\cdots d}$.

Similar results hold good for the extensions of the affine connection $\Gamma^i_{\alpha\beta}$. For example, the set of functions

$$\frac{\partial^{k*}\Gamma^i_{\alpha\beta}}{\partial y^{\gamma_1}\cdots\partial y^{\gamma_k}}, \qquad (k \leqq r),$$

evaluated at the origin of a geodesic coordinate system y^i of order $r+1$ define the kth *normal tensor* $A^i{}_{\alpha\beta\gamma_1\cdots\gamma_k}$. Thus the replacement theorems for affine differential invariants† of order r can be proved on the basis of geodesic coordinates of order $r+1$ without any hypothesis as to the existence of partial derivatives of the components of affine connection of higher order than the rth. Finally the replacement theorems for metric differential invariants‡ of order r can be proved on the basis of geodesic coordinates of order r without any hypothesis as to the existence of partial derivatives of the fundamental metric tensor $g_{\alpha\beta}$ of higher order than the rth.

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* The proofs of these statements are left for the reader since they are obtained by an obvious modification of Veblen's normal coordinate methods. See Veblen and Thomas, loc. cit., pp. 569–573.

† T. Y. Thomas and A. D. Michal, Annals of Mathematics, vol. 28 (1927), pp. 196–236

‡ T. Y. Thomas and A. D. Michal, Annals of Mathematics, vol. 28 (1927), pp. 631–688.