ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THIS SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume.* Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

217. Professor E. T. Bell: *A correspondence between irregular fields.*

Dickson extended the theory of numbers by a correspondence between fields (this Bulletin, vol. 23 (1916), p. 109). Correspondence is here defined for irregular fields, and is applied to such fields whose elements are respectively the (infinite) set of all numerical functions, and a certain infinity of power series with radius of convergence unity. The regular elements (those having reciprocals) in the second irregular field are those power series of the kind defined with constant term different from zero. The simplest example of multiplication in this irregular field is the process by which, when legitimate, a Lambert series is derived from a given power series. Examples can be constructed showing that the restriction to the unit circle as the region of convergence is necessary as well as sufficient. (Received April 5, 1930.)

218. Professor E. T. Bell: *The real unit segment as a number field.*

In a general discussion of the theory of unique decomposition (multiplicative, additive, etc.) into indecomposable elements, it becomes necessary to devise an algebra, arithmetic and analysis for real vectors with $n>3$ components. This problem can be reduced to the situation described in the title. The set of all real numbers in the closed interval $[-\infty,+\infty]$ is put in (1,1) correspondence with $[0,1]$, so that $-\infty, 0, 1, +\infty$ correspond respectively to $0, 1/2, 3/4, 1$, and the rational operations in $[0,1]$ are then defined in abstract identity with those for $[-\infty,+\infty]$ as in analysis. Order relations are introduced in an obvious manner. To the rational integer m in $[-\infty,+\infty]$, corresponds in $[0, 1]$ $m'=1/2+(1/\pi)\tan^{-1} m$, $|\tan^{-1} m|\leqq\pi/2$, and the "rational integer" m' in $[0, 1]$ is "divisible arithmetically" by the rational integer n' in $[0, 1]$ when and only when m is arithmetically divisible by n in $[-\infty, +\infty]$. (Received April 5, 1930.)

219. Professor E. T. Bell: *Numbers of representations in certain senary quadratic forms.*

---

*See pp. 1–2 and p. 45 of the January issue.

The form $x^2+ay^2+bz^2+cw^2+du^2+et^2$ is denoted by $(1, a, b, c, d, e)$. It is shown that the number of representations of any integer n in each of the forms (1, 1, 1, 1, 1, 4), (1, 1, 1, 1, 4, 4), (1, 1, 1, 4, 4, 4), (1, 1, 4, 4, 4, 4), (1, 4, 4, 4, 4, 4), is a polynomial in the real divisors of n alone, with the single exception of the case $n\equiv 1$, (mod 4). If $n \equiv 1$, (mod 4), then the number of representations is a polynomial in the divisors of n plus a numerical constant times the number of representations of n in the form (1, 1, 1, 1, 1, 16). The author has not succeeded in proving whether or not the last is reducible to a divisor function. (Received April 5, 1930.)

220. Professor E. T. Bell: *Analytic functions in the irregular field of all numerical functions.*

An irregular field ϕ differs from an abstract field F only in the presence in ϕ of an infinity of elements having no reciprocal in ϕ, instead of the unique such element zero in F. In the author's *Algebraic Arithmetic* (Colloquium Publications of this Society, vol. 7, 1927) was summarized his previous work on the algebraic aspects of the irregular field of all numerical functions. The algebraic theory there developed is complete. In this paper the theory is extended to essentially infinite processes in the same irregular field, and a complete theory of such is given. As an interesting detail, the functional equations for the elementary functions and for the theta and elliptic functions have, in the irregular field, solutions reducible to simple closed forms. The formal properties of the new functions are abstractly identical with those of the corresponding analytic functions in the real or complex domain. (Received April 5, 1930.)

221. Professor J. V. Uspensky: *On the reduction of the indefinite binary quadratic forms.*

According to Hermite the reduction theory of indefinite binary quadratic forms can be based on consideration of integers giving successive minima of a positive form $\xi^2+\lambda\eta^2$ where λ is a variable positive parameter and ξ and η represent the conjugate linear factors of an indefinite binary quadratic form. However, the reduced forms obtained by this method differ from those of Gauss. In order to obtain Gauss' reduced forms one must consider a nonhomogeneous function $F=|\xi/\eta|+\lambda|\eta|$ and its successive minima as a substitute for Hermite's positive form. (Received April 5, 1930.)

222. Professor F. S. Nowlan and Mr. Ralph Hull: *Sets of integral elements of certain rational Dickson algebras.*

Sets of integral elements are determined which belong to certain rational division algebras of the type discussed in Dickson's *Algebras and their Arithmetics*, p. 66. The basal units of the algebra are y^ix^j, $(i, j=0, 1, 2)$, where x satisfies the irreducible cyclic cubic $x^3-3x+1=0$, and $y^3=\delta$, with δ a rational integer. We write $\delta=\eta\epsilon$, where η has no rational prime factors other than 3 or those of the forms $9m\pm 1$, and where the rational prime factors of ϵ are of the forms $9m\pm 2$ and $9m\pm 4$, with one factor occurring to a power which is not a multiple of 3. We add the further restriction that ϵ is itself of one of the forms $9m\pm 2$ or $9m\pm 4$. For each such algebra there is a unique set of integral elements which contains the original nine basal units. These original basal units do not, however, constitute a basis. (Received March 24, 1930.)

223. Dr. A. A. Albert: *On direct products, cyclic division algebras, and pure Riemann matrices.*

In this paper certain theorems on the direct products of normal simple algebras are secured, and a necessary and sufficient condition that a direct product of any two division algebras be a division algebra is found. The theory of the representation of a normal division algebra in n^2 units over F as an algebra of m-rowed square matrices with elements in F is discussed, and it is proved that such a representation is possible if and only if n^2 divides m. The theory of the structure of cyclic division algebras is studied, and the problem reduced to the case of algebras of orders p^{2e}, p a prime. Also certain necessary conditions that a cyclic algebra be a division algebra are discussed. Finally the above results are utilized to prove two extremely important results in the theory of pure Riemann matrices: *The multiplication index of a pure Riemann matrix of genus p is a divisor of $2p$; when the multiplication algebra of a pure Riemann matrix is a known normal division algebra, its order is a power of two.* (Received March 12, 1930.)

224. Mr. Leonard Carlitz: *On Galois fields of certain types.*

In this paper are deduced relationships between Galois fields of prescribed construction and some of their sub-fields. Relations between the zeta functions and the discriminants of the various fields are set up. It is not here found necessary (as in the work of Artin, Herglotz, and Pollaczek) to make use of the Hecke functional equation for the zeta function of an arbitrary field. From the zeta and discriminantal relations it is possible to express a quotient of class numbers in terms of a corresponding quotient of regulators. In a particular case (the group of the Galois field is the group of a solvable equation of prime degree) the quotient of regulators is examined and shown to lie between bounds of a simple sort. (Received March 25, 1930.)

225. Mr. Charles Wexler: *On real quadratic fields.*

By means of new and rather simple methods, suggested by computing the classes in all fields up to 1,000, the following theorems concerning real quadratic fields are proved. The norm of the fundamental unit in an infinite number of fields is equal to -1. If $D=p$ or $2p$, where $p\not\equiv 1 \pmod 4$ is prime, then the field defined by $D^{1/2}$ has an odd number of classes. A necessary and sufficient condition that such a field be simple is given. As an interesting special case, it is proved that if D is prime or twice a prime, and if it can be written in the form $n^2\pm 2$, then a necessary and sufficient condition that the field defined by $D^{1/2}$ be simple is that $D-a^2=p$ or $2p$, p a prime, for all $a<D^{1/2}$. In general, if D is composite, necessary and sufficient conditions that each genus contain but one class are given. (Received March 27, 1930.)

226. Professor R. L. Jeffery: *The uniform approximation of a summable function by means of step-functions.*

Let $f(x, y)$ be bounded and measurable on a bounded domain R and let S be a measurable set on R. We are concerned with the existence of a function $K(x, y)$ constant on each of a finite number of rectangles, and such that $\int_{S_x}|f-K|dx<\epsilon$ uniformly in y, and $\int_{S_y}|f-K|dy<\epsilon$ uniformly in x, where S_x and

S_y are the respective linear sections of S. Let F be bounded and measurable on a bounded n-dimensional domain R. Let ω_p be a cell with equal sides containing a point p of R. Let $I_p{}^\omega$ be the ratio $\int_{\omega_p} Fdx : m\omega_p F(p)$. Let E be the part of R for which $I_p{}^\omega$ converges to unity as ω_p converges to zero. Then $mE = mR$, and E contains a subset $E_{\epsilon\eta}$ with $mE - mE_{\epsilon\eta} < \epsilon$ at each point of which F has saltus $< \eta$ relative to $E_{\epsilon\eta}$. It is then sufficient for the existence of $K(x, y)$ that for each η and λ there exists δ such that both $m(CE_{\epsilon\eta} \cdot S_x)$ and $m(CE_{\epsilon\eta} \cdot S_y)$ are less than λ for all x and y, provided $\epsilon < \delta$. (Received March 26, 1930.)

227. Professor I. M. Sheffer: *Note on point transformations and non-analytic functions.*

A non-analytic function $f(z)$ generates a point transformation which is in general non-(directly) conformal; angle is thus distorted. Hedrick, Ingold, and Westfall have studied this aspect, and have arrived at interesting results; they derive the "characteristic lines" and set up a classification of non-analytic functions. The present paper starts from the same hypothesis of non-conformality. To each direction dz at a point we associate a direction δz such that the angle between dz and δz is preserved under transformation. Such pairs we term *dual directions*. There is a unique pair of orthogonal dual directions at a point; these give the "characteristic lines." There is a unique pair of coincident dual directions: the *contact directions*. The function $|f(z)|$ has its extremals along the characteristic lines (as Hedrick, Ingold, and Westfall have shown) and amp $f(z)$ has its extremals in the contact directions. These directions are interpreted on the derivative circle, and the position of dual directions found. Finally, we consider derivative conditions which serve to determine $f(z)$. (Received March 24, 1930.)

228. Professor H. T. Davis: *The homogeneous case of the linear differential equation of infinite order with polynomial coefficients.*

The problem studied is that of determining conditions under which solutions exist for the differential equation $\sum_{i=0}^{\infty} A_i(z)x^i \rightarrow u(x) = 0$, where the $A_i(z)$ are functions having Taylor's expansions about the origin and z is the differential operator d/dx. Two classes of solutions are discussed: (1) those of bounded degree (Stufe) and (2) those of unbounded degree. By degree we mean the limit $L = \lim_{n\rightarrow\infty} |u^{(n)}|^{1/n}$. The Pincherle-Bourlet generatrix equation is made central to the discussion and the solutions appear in the form of Laplace integrals. In the bounded class the theorem of Perron (Mathematische Annalen, vol. 84 (1921), pp. 31–42) is derived. In the unbounded class Borel summability is employed and an extension of Cauchy's integral formula for the case of fractional index is found useful. When $p = 1$, the regions of integration of the Laplace integrals are discussed for the cases (1) where the positive powers term of the Laurent expansion of $A_0(z)/A_1(z)$ in the outer annulus is algebraic, and (2) where it is transcendental. (Received March 20, 1930.)

229. Mr. W. M. Cady: *An extension of the classical algebra and calculus.*

A generalization of the ordinary modes of combination of complex numbers, viz. addition, multiplication, and exponentiation, leads to a denumerable infinity of other modes. The algebra embraced by any two consecutive modes is commutative, associative, and distributive. In the present paper the modes above exponentiation are investigated; certain difficulties with the modes below addition have prevented work in that direction. It is shown that all numbers (with certain deletions) form a group for any mode of combination, and that these groups are isomorphic to each other. An extension of the algebras thus developed leads to a generalization of the integral calculus, whereby all the elements of a continuum may be combined under any mode. In particular, the calculus corresponding to multiplication is discussed, and certain theorems are proved for it which are analogous to theorems in the ordinary calculus. (Received March 27, 1930.)

230. Mr. R. P. Agnew: *The behavior of bounds and oscillations of sequences of functions under regular transformations.*

In this paper is discussed the transformation of sequences of functions by regular linear transformations with triangular matrices whose application to sequences of constants has been extensively treated. The first chapter is on the behavior of ultimate bounds of real sequences under real transformations. The second and third chapters consider continuous oscillation, continuous convergence, uniform oscillation, and uniform convergence of complex and real sequences under complex and real transformations. In each case, bounds, oscillations, and convergence are considered (1) over the set as a whole, (2) at a single point or limit point of a set, and (3) at all points and limit points of a set. The fourth chapter is on the behavior of mean-square oscillation and convergence in the mean of complex and real sequences under complex and real transformations. (Received March 25, 1930.)

231. Mr. Leonard Carlitz: *On a class of finite sums.*

Generalizing a well known method for the evaluation of $\sum_{i=1}^{m} i^n$ we show in this paper how to sum $\sum_{i=1}^{m}[i(i-1) \cdots (i-\lambda+1)]^n$. The general term is expressed as a sum of factorials. The coefficients involved form arrays of numbers that can be calculated from a recursion formula; for $\lambda=1$ the array is the array of Stirling numbers. The expression for the sum contains $\lambda(n-1)+1$ terms. When λ is even, the method may be modified in such a way as to reduce the number of terms to approximately one-half. (Received March 25, 1930.)

232. Dr. A. A. Albert: *The structure of matrices with any normal division algebra of multiplications.*

Let B, a normal division algebra of n^2 units over F, contain a quantity a of grade n with respect to F and $\alpha_1, \cdots, \alpha_n$ as scalar roots of its minimum equation. Suppose that m is an integer divisible by n^2 so that B is representable by an algebra of m-rowed square matrices with elements in F and in particular

$m = nm'$. The author has proved the existence of certain n-matrices P_j which are m'-rowed square non-singular matrices with elements in $F(\alpha_1, \cdots, \alpha_n)$, and that when F is a field of complex numbers and ω a complex matrix with $p'n$ rows and m columns, the matrix ω has B as an algebra of multiplications if and only if ω is isomorphic in F to $\| tP_j\alpha_j^{k-1} \|$, $(j, k = 1, \cdots, n)$, where t has p' rows and m' columns. This result is applied to give a complete determination of the structure of any pure Riemann matrix. (Received March 12, 1930.)

233. Mr. Hassler Whitney: *Note on central motions.*

Birkhoff (Göttinger Nachrichten, 1926, pp. 81–92) states that in the two-dimensional analytic case the second set of non-wandering motions M_2 is identical with the central motions M_r. This note proves the same conclusion for the case of a one-to-one continuous transformation on the surface of the sphere. (Received March 22, 1930.)

234. Professor Maurice Fréchet and Dr. J. A. Shohat: *A proof of the generalized second limit-theorem in the theory of probability.*

The above-mentioned proof is based on the following lemma. Given a sequence of laws of total probability $F_n(x)$ $(n = 1, 2, \cdots)$ having moments of all orders. Assume that the moments of order k of this sequence are uniformly bounded for any given $k = 0, 1, 2, \cdots$. Then at least one subsequence $F_{n_p}(x)$ exists which approaches a limiting law of total probability $F(x)$ for any x in $(-\infty, \infty)$, so that the kth moment of $F_{n_p}(x)$ approaches the kth moment of $F(x)$ $(k = 0, 1, 2, \cdots)$, Two features, it seems, deserve attention. First, the generality of the statement, including the classical theorem as a very special case. Second, the method of proof, which uses neither characteristic functions nor continued fractions, but is based on a well known theorem on monotonic functions. (Received March 28, 1930.)

235. Mr. Nathan Howitt: *The eliminant in electric circuit theory.*

The Kirchoff equations of the electric circuit can be obtained from the inductance, capacity, and resistance quadratic forms and Lagrange's equations. The impedance function is the quotient of the determinant of the coefficients of the currents and the minor of the element in the first row and first column, and for a finite network is the quotient of two polynomials. The coefficients of these polynomials are obtained directly from the matrices of the coefficients of the above quadratic forms. The conditions for the preservation of the form of the impedance function are given in terms of the eliminant of these two polynomials. Short circuit corresponds to the vanishing of the eliminant. By removal of elements of the network subject to the above conditions, the so-called canonical forms, obtained by Foster by partial fraction expansion and by Cauer by continued fraction expansion, result. The conditions that a function having the form of an impedance function be an impedance function of a physical network are given in terms of the eliminant, and the equivalence equations for the invariance of the impedance function are generalized for n meshes. (Received April 14, 1930.)

236. Professor J. F. Ritt: *Manifolds of functions determined by systems of algebraic differential equations.*

We consider any system of algebraic differential equations in one independent variable x, and in n dependent variables $y_1, \cdots, y_n$, the coefficients in the equations being analytic functions belonging to a given field. A system is called irreducible if, given two polynomials G and H in the y's and their derivatives, with coefficients in the given field, such that GH vanishes for all solutions of the system, then either G vanishes for all solutions or H does. It is shown that every system is equivalent to a finite number of irreducible systems. The structure of an irreducible system is then investigated. A single equation, called the resolvent of the irreducible system, is found, in terms of the general solution of which the complete solution of the irreducible system can be expressed by rational processes. The results obtained create a perfect analogy between the notion of the complete solution of a system of algebraic differential equations and the notion of algebraic function of several variables. (Received April 17, 1930.)

237. Mr. R. P. Agnew: *On uniform summability of sequences of continuous functions.*

It has been shown by D. C. Gillespie and W. A. Hurwitz that any bounded real sequence $\{s_n(x)\}$ of continuous functions, defined over a closed compact set in metric space, which converges to a continuous function $s(x)$, is uniformly summable to $s(x)$ by a regular transformation. The transformations used were transformations with square matrices which were constructed in terms of the elements of the $\{s_n(x)\}$ sequences considered; hence different transformations were used to effect the uniform summability of different sequences. In the present note it is shown that no one regular transformation of a very general type will suffice for the uniform summability of all sequences of continuous functions which converge over an infinite set to a continuous limit function. (Received March 31, 1930.)

238. Mr. H. L. Garabedian: *On the relation between certain methods of summability.*

In this paper a comparison is made between Cesàro's definition of summability and a set of definitions which are in the main of the Lindelöf type (see Lindelöf, Journal de Mathématiques, (5), vol. 9 (1903), p. 213). In particular, it is proved that every series summable (Cr), r integral and arbitrary, is also summable by the methods of LeRoy and Mittag-Leffler. Moreover, a sequence of Dirichlet's series definitions which include (Cr) summability is exhibited. (Received April 8, 1930.)

239. Professor Edward Kasner: *Geometry of element transformations.*

In the American Journal of Mathematics for 1910, the author discussed arbitrary transformations of differential elements, with reference to the number of unions which are turned into unions. For lineal elements in the plane (x, y, p), where p is the slope, ∞^2 unions are preserved. If there are more we have necessarily a contact transformation. In the present paper are studied curves which are termed *isogonal series*, such a series being defined as ∞^1 elements which make a constant angle α with the curve obtained by joining their points. (Dually we may also define equitangential series.) In this way ∞^3 curves are obtained, ∞^2 for each value of α. For a given element (x,y,p) there are ∞^1 such curves, and it is proved that the center of curvature varies in homographic correspondence with tan α. The differential equation of the triply infinite familyis obtained, and some geometric consequences are stated. Extensions to lineal and surface elements in space are much more complicated. (Received April 18, 1930.)

240. Mr. J. H. C. Whitehead: *The representation of projective spaces.*

This paper deals with the representation of a projective space of paths P_n, by means of an affine space, a_{n+1}, the projective connection being regarded as an affine connection for A_{n+1}. It is found that A_{n+1} admits a congruence of paths, defined by a vector which satisfies the equations $\xi^\alpha;\ \beta=\delta_\beta{}^\alpha$, and which defines a group of affine collineations for A_{n+1}. These paths all meet in a point, and, as in the flat case, each one represents a point of P_n. Assuming the affine theory, a geometric method of obtaining projective normal coordinates, z^i, is given, and it is shown that they are related to the affine normal coordinates for A_{n+1} by the equations $y^i=z^i$, $y^0+1=e^{z^0}$, where z^0 is the normal factor. It is also shown that if there exists a transformation T of the $(n+1)$ variables $x^0, \cdots, x^n$ which carries one of two projective connections into the other, then they are also equivalent under transformations of the form $x^0=x^0+\log\rho(x^1,\cdots,x^n)$, $x^i=x^i(x^1,\cdots,x^n)$, though this need not necessarily be T itself. Finally a process is given by which a projective connection may be normalized to obtain a projective connection in the sense of T. Y. Thomas, and it is shown that all the results obtained are applicable to the theory of these projective connections. (Received April 18, 1930.)

241. Professor B. F. Kimball: *Note on the Gauss-Bonnet formula of differential geometry.*

The author gives a simple proof of the elegant Gauss-Bonnet formula by using surface vectors. This proof is an extension of the method used by C. E. Weatherburn to the more general case of any simply connected region containing only regular points of the surface, the boundary to be rectifiable with a continuously turning tangent. (Received March 28, 1930.)

242. Professor Malcolm Foster: *Theorems on rotated and inverted congruences.*

When the unit sphere is referred to any isothermal system, each line of a rectilinear congruence is associated with that point on the sphere at which the normal has the same direction, or with similar points on the two adjoint minimal surfaces determined by the given isothermal system. Thus a congruence is defined by the point (a, b) in which any line pierces the xy-plane of the moving axes at the associated point. This paper is concerned with those congruences derived by an inversion of the point (a, b) relative to the unit circle with center at the associated point and lying in the tangent plane, or those congruences derived by a rotation of (a, b) about the corresponding normal. (Received March 24, 1930.)

243. Professor T. R. Hollcroft: *Degenerate algebraic manifolds.*

The characteristics of a manifold M of order n and dimension r, M consisting of α manifolds of dimension r and orders n_i such that $\Sigma n_i = n$, are found. It is shown that the intersections of the component manifolds of M of dimension r taken j at a time, $2 \leqq j \leqq r+1$, are j-fold $(r-j+1)$-dimensional manifolds on M and on all singular manifolds of M of higher dimension than the one specified, and play the same role on M as proper multiple manifolds on a proper manifold. Bertini proved that the genus of a degenerate plane curve consisting of α components of genera p_i is $\Sigma p_i - \alpha + 1$. This is shown to hold for curves in any space and similar relations are found for surfaces and three-dimensional manifolds in any space, which leads to the following general theorem: If a degenerate manifold M of dimension r in k dimensions, $k \geqq r+1$, consists of α non-degenerate manifolds, each of dimension r and genus p_i, $i = 1, 2, \cdots, \alpha$, the genus p of M is given by the formula $p = \Sigma p_i \pm (\alpha - 1)$, in which the plus sign occurs when r is even and the minus sign when r is odd. (Received March 28, 1930.)

244. Dr. A. B. Brown (National Research Fellow): *Coalescence of parts of a complex.*

Relations are found involving the changes in Betti numbers that occur when a number of homeomorphic parts of a complex are made to coincide. In an earlier paper the author treated the case where two parts on unconnected complexes are made to coincide, and obtained more detailed relations which have important applications. In a special case of the latter case, formulas are obtained in the present paper by use of which we can find the Betti numbers of any one of the four complexes involved if we know the Betti numbers of the other three. (Received April 18, 1930.)

245. Dr. J. H. Roberts (National Research Fellow): *A note concerning cactoids.*

A cactoid M is a bounded continuous curve lying in space of three dimensions and such that (a) every maximal cyclic curve of M is a simple closed surface and (b) no point of M lies in a bounded complementary domain of any subcontinuum of M. Now it is known that there exists a bounded acyclic continuous curve C such that every bounded acyclic continuous curve is homeomorphic with a subset of C. G. T. Whyburn has shown that with respect to its cyclic elements every continuous curve is acyclic. Moreover, the cyclic elements of a cactoid are either points or spheres. Thus this question arises: *Does there exist a cactoid C such that every cactoid is homeomorphic with a subset of C?* The object of the present paper is to answer this question negatively. (Received March 27, 1930.)

246. Professor D. W. Woodard: *Sets strongly homeomorphic with the interior of a plane circle.*

A set M_1 is said to be strongly homeomorphic with a set M_2 provided (1) M_1 is homeomorphic with M_2 and (2) $\overline{M_1}$ is homeomorphic with $\overline{M_2}$. Necessary and sufficient conditions that a set be homeomorphic with the interior of a plane circle are established. It is shown that a characterization of the sphere may be realized by a simple modification of one of the conditions. (Received March 28, 1930.)

247. Dr. Leo Zippin: *Conical accessibility and the topologic sphere.*

A continuous curve in euclidean three-space is said to be conically accessible from one of its complementary domains, D, provided that if K is a simple closed curve of the continuous curve and on the boundary of D, then K bounds a two-cell of D. It is proved that a continuous curve C which separates three-space into two domains of each of which it is the complete boundary, and from each of which it is conically accessible, is necessarily a topologic sphere, if it is compact, and a cylinder tree if it is non-compact. (Received March 14, 1930.)

248. Mr. V. W. Adkisson: *Cyclicly connected continuous curves whose complementary domain boundaries are homeomorphic, preserving branch points.*

This paper classifies certain cyclicly connected continuous curves in euclidean space of two dimensions. Among the curves M considered are those that contain only a finite number of simple closed curves, and are such that any two complementary domain boundaries are homeomorphic, preserving branch points. Such curves are classified according to the number and orders of the branch points on each complementary domain boundary. First a necessary and sufficient condition is established that a continuous (1, 1) correspondence of M (on a sphere S) into itself be extendable to S; then the curves M are classified. We prove also that any continuous (1, 1) correspondence between any two complementary domain boundaries of M can be extended to M, and also to S, for the curves in which each complementary domain boundary contains exactly three branch points. Finally, it is shown that for any one of the curves M any continuous (1, 1) correspondence sending the curve into itself can be extended to the sphere S. (Received March 28, 1930.)

249. Professor R. G. Putnam: *Note on components and constituents of open sets.*

In this note it is proved that a necessary and sufficient condition that the components of every open subset of an open set M should be open is that M be locally connected. It is also shown that this same condition must be satisfied when the constituents of the open subsets of M are open. A condition sufficient to insure that the constituents of the open subsets of M be open is obtained. (Received March 25, 1930.)

250. Professor G. T. Whyburn: *Potentially regular point sets.*

A separable metric space R is called *potentially regular* if for each point P of R there exists a monotone decreasing family of closed neighborhoods of P whose common part is exactly P and each of which has only a finite number of boundary points. If R is connected, it is potentially regular if and only if every two points of R can be separated by a finite number of points. Let R be any connected and potentially regular space; then there exists a biunivalued and continuous transformation T of R into a separable, metric space R^* which is connected and regular in the Menger-Urysohn sense; furthermore, the property of a finite set to separate two given points of R is invariant under the transformation T. (Received March 12, 1930.)

251. Professor G. T. Whyburn: *Concerning the structure of regular curves.*

Let M be any regular curve (in the Menger-Urysohn sense), and let K be any subset of M which contains the (countable) set Q of all local separating points of M of order >2 and which is dense on every free arc in M; then every point p of M is a regular point of M relative to K, and, indeed, the order of p in M relative to K is equal to its true order relative to M itself. The following theorem concerning additions of regular curves is also proved: In order that the sum $M+N$ of a given regular curve M and an arbitrary regular curve N should be regular, it is necessary and sufficient that the closed envelope $\bar{H}$ of the set H of all ramification points of M should be punctiform. This latter theorem gives a solution to a problem proposed by Menger. (Received March 12, 1930.)

252. Professor G. T. Whyburn: *Concerning irreducible ϵ-separations, locally connected spaces, and accessible plane continua.*

The following theorem yields solutions to two problems proposed by Urysohn (Verhandelingen der Akademie te Amsterdam, 1927, No. 4, p. 144). In order that a continuum M be connected im kleinen it is necessary and sufficient that for each point x of M and each $\epsilon>0$, there exist an irreducible ϵ-separation of x in M. A separable, metric, and connected space R is locally connected if and only if every cutting of R between two arbitrary subsets A and B of R contains an irreducible cutting between A and B; also when and only when there exists an irreducible cutting of M between every two separated subsets A and B of R. Also a generalization of a result of Mazurkiewicz on accessible points of indecomposable continua is obtained. (Received March 12, 1930.)

253. Professor Solomon Lefschetz: *Transformations of compact spaces and their fixed points.*

In the present paper the author extends to continuous transformations of absolutely general compact metric spaces the coincidences and fixed point formulas given for manifolds in papers in volumes 28 and 29 (1926–27) of the Transactions of this Society. As an important corollary it follows that a continuous single-valued transformation of a compact metric space whose Betti numbers are those of the Hilbert parallelotope always has a fixed point. The proofs will appear as part of Chapter VII of the author's forthcoming Colloquium Lectures. (Received April 18, 1930.)

254. Professor A. H. Copeland: *Point set theory applied to the random selection of the digits of an admissible number.*

Admissible numbers enable us to answer certain questions in the theory of probability which could not be handled without some such device. An admissible number belonging to the set $A(p)$ exhibits all of the properties of an event having the probability p. For example, the success ratio approaches p as a limit and the trials of the event are independent. In this paper we shall consider whether these properties are still exhibited if we form a new event (or new number) by selecting arbitrarily a sub-group of trials (or digits of the admissible number). Obviously this will not be the case for every such choice of the digits. However, it will be proved that in general the new number will have the same properties as the original number. (Received March 13, 1930.)

255. Professor C. C. MacDuffee: *The discriminant matrix of a semi-simple algebra.*

If $\mathfrak{A}$ is a semi-simple algebra, a basis can be so chosen that e_1 is the principal unit and the first discriminant matrix is diagonal, the diagonal elements being denoted by $g_1, g_2, \cdots, g_n$. For this normalization, the constants of multiplication c_{ijk} are in the relation $g_k c_{ijk} = g_i c_{jki}$. By using this normal form, several important properties of semi-simple algebras are established, e.g., that the first and second discriminant matrices are identical for all bases, and likewise the first and second characteristic functions. Theorems concerning the discriminant matrix of a direct sum, direct product, and complete matric algebra are also given. (Received March 25, 1930.)

256. Mr. C. W. Mendel: *A new characterization of the surface of Veronese.*

This paper considers plane curves and the so-called principal curves on surfaces immersed in five-dimensional space. It contains, among other things, a new property which characterizes the surface of Veronese, namely, that the only surface which is immersed in five-dimensional space and which has on it a two-parameter family of plane curves is the surface of Veronese. Moreover, there is given a new and simple derivation of a property which Segre proved to be characteristic of this surface, namely, that the only non-developable surface which is immersed in five-dimensional space and on which the principal curves are indeterminate is the surface of Veronese. (Received March 19, 1930.)

257. Professor W. L. Ayres: *A generalization of the Scherrer fixed-point theorem.*

It is shown that if T is a homeomorphism of the compact locally connected continuum M into a subset of itself, then M contains a cyclic element C such that $T(C)$ is a subset of C. The following two interesting results are corollaries to this theorem: If each maximal cyclic set of M is an n-dimensional simplex (n may vary for different maximal cyclic sets), then every such homeomorphism T has a fixed point. If M is a plane bounded locally connected continuum, M does not separate the plane, and T is a homeomorphism of M into a subset of itself, then T has a fixed point. The results solve a problem proposed by C. Kuratowski (Fundamenta Mathematicae, vol. 14, p. 139, footnote 2). (Received March 27, 1930.)

258. Dr. P. M. Swingle: *A generalization of biconnex sets.*

Two definitions of biconnex sets by B. Knaster and C. Kuratowski, *Sur* *les ensembles connexes* (Fundamenta Mathematicae, vol. 2, pp. 214, 216) are generalized and the resulting sets are shown to be equivalent. A number of theorems are proved concerning the set thus defined and related sets. (Received March 28, 1930.)

259. Professor E. P. Lane: *Hypergeodesic mapping of a surface on a plane.*

The purpose of this paper is to study a projective analogue of the metric problem of Beltrami, to map a surface of ordinary space on a plane so that the geodesics on the surface correspond to the straight lines in the plane. The geodesics are replaced by hypergeodesics, which are defined projectively by a differential equation of the same form as the equation of the geodesics. Fubini's canonical form of the differential equations of a surface, and Wilczynski's theory of plane nets are employed to construct a general theory of this kind of hypergeodesic mapping. Then three special cases of hypergeodesic mapping are studied in some detail. Green's congruentially associated net of a given plane net is connected with the theory. A new two-parameter family of plane curves covariant to a given plane net is defined and used in a second kind of hypergeodesic mapping of a surface on a plane. (Received March 21, 1930.)

260. Dr. Lincoln La Paz: *A characteristic normal form for the Euler equations of regular problems of the calculus of variations with prescribed transversality conditions.*

The most general integrand function f for a regular problem of the calculus of variations with a prescribed (admissible) set of transversality coefficients $T_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$, $i=1, \cdots, n$, has been determined by the writer. In the present paper a characteristic normal form for the Euler equations of such problems is obtained. It is shown that $y_k''=D_k/D$, where D is the Jacobian of the T_i with respect to the y_j' and where D_k is a determinant derived from this Jacobian by change of the kth column. In the special case in which transversality is orthogonality and in which $n=2$ the resulting normal system is reducible to the pair of equations known to be characteristic of a *natural family* (Kasner). Application is made to inverse problems of the calculus of variations. (Received March 28, 1930.)

261. Dr. W. T. Reid (National Research Fellow): *Expansion problems associated with a system of integral equations.*

This paper considers the vector integral equation $y(x)=\lambda\int_a^b K(x; s)y(s)ds+f(x)$, where $y(x)\equiv(y^{(\alpha)}(x))$ and $f(x)\equiv(f^{(\alpha)}(x))$ are vectors with n components and $K(x; s)\equiv(K_{ij}(x; s))$ is a matrix with n rows and columns. The vector method is used throughout in determining the form of the Fredholm determinant and resolvent matrix for the vector equation. It is shown that when the equation is definitely self-adjoint according to a suitable definition, the characteristic numbers are all real and infinite in number. There are established expansion theorems for certain classes of functions in terms of the characteristic solutions of the definitely self-adjoint equation. The definitely self-adjoint equation includes as a special case the symmetric system in which $K_{ij}(x; s)=K_{ji}(s; x)$, $i, j=1, 2, \cdots, n$, and the kernel matrix is closed. It also includes as a special case the system of integral equations to which a differential system consisting of n ordinary linear differential equations of the first order together with two-point boundary conditions that is definitely self-adjoint in the sense defined by Bliss (Transactions of this Society, vol. 28 (1926), pp. 561–584) may be reduced by the introduction of the Green's matrix for the differential system. (Received March 24, 1930.)

262. Mr. I. S. Sokolnikoff: *On solution of the torsion problem for a polygon with reentrant angles.*

The solution of $\nabla^2\phi=0$ with the boundary condition $\phi=(x^2+y)^2/2$ is obtained for the T-section with an infinite flange and web. The method of solution is different from that of Trefftz and Kötter, both of whom solved the problem for an L-section, which involves only one reëntrant angle. The boundary of the T-polygon is transformed by Schwartz's transformation into the axis of reals, and the interior of the polygon into the upper half-plane. The problem is reduced to the solution of the potential problem for a half-plane with prescribed values along the axis of reals; its solution is effected by Poisson's integral in which the path of integration is transformed conformally into the real axis. The resulting integral together with the equations of transformation constitutes the solution of the problem in parametric form. The method is of sufficient theoretical generality to make it applicable to the solution of $\nabla^2\phi=0$ with prescribed values along the boundary of any rectilinear polygon. (Received March 25, 1930.)

263. Dr. Wilhelm Maier: *Elementary properties of the t_ν-functions.*

In constructing an analogue of Bernoullian polynomials corresponding to the imaginary quadratic field, we are led to the following expressions:

$$\lim_{n\to\infty}\sum_{h,k}^{0<h^2+k^2<n} e^{2\pi i(xh+yk)}/(\omega_1 h+\omega_2 k)^\nu = t_\nu(x,y)=t_\nu,$$

where the "periods" ω_1, ω_2 do not vanish together, and $0<|\Im(\omega_2/\omega_1)|$; $0<x, y<1, \nu=1, 2, \cdots$. There are non-linear recursion formulas by which the whole set depends rationally on t_1, t_2, t_3, and a cubic identity which connects t_1, t_2, t_3, so that the study of these functions reduces to studying t_1, t_2. If $m, n=1, 2, \cdots$; $\alpha, \beta=0, \cdots, n-1$; $0<\alpha+\beta$; $t_n(0, 0)=\tau_n$; there exists an equation of degree n^2, whose coefficients a_ν are rational in τ_4, τ_6, t_m, so that $\sum_{\nu=0}^{n^2} a_{n^2-\nu}t_m{}^\nu(x\alpha/n, y\beta/n)=0$. Since $t_n(x, y)=t_n(x+1, y)=t_n(x, y+1)$, we are enabled to express elliptic functions rationally in terms of t_ν-functions. For instance, as pointed out by Siegel, $t_1^2+2t_2=\wp(\omega_1 y-\omega_2 x)$. All the theory of elliptic functions and, in particular, the algebraic relations between functions of divided argument as given by N. Abel may be based on the present theory. (Received March 10, 1930.)

264. Professor K. P. Williams: *The constants of the disturbing function.*

A systematic development by means of contour integrals is given for the constants $c_n^{(i,j)}$ that occur in Newcomb's development of the disturbing function. Recursion relations are obtained that both are very accurate and rapid to use, and greatly lessen the labor of calculation. The few constants that are obtained directly are given by rapidly converging series similar to those given by Brown. (Received March 27, 1930.)

265. Mr. E. W. Miller: *Concerning subsets of a continuous curve S which lie on an arc of S.*

Conditions that a closed, bounded subset of the euclidean plane lie on an arc of the plane are due to R. L. Moore and J. R. Kline. The present paper considers the corresponding problem when the imbedding space is any continuous curve in euclidean n-space. In the course of study of the Moore-Kline conditions, it is shown that the only plane closed sets for which they represent a solution of the problem are the plane itself and the simple continuous curves. Among the results obtained is a theorem reducing the problem in euclidean n-space to the case where the continuous curve is cyclicly connected. On the basis of this theorem, the problem is solved for acyclic continuous curves, and continuous curves which are boundaries of domains complementary to plane continuous curves. For a plane continuous curve S which fails to cut the plane, it is shown that a closed totally disconnected subset F lies on an arc of S if and only if for every cyclic element, C, of S at most two components of $S-C$ contain points of F. (Received March 28, 1930.)

266. Professor V. G. Grove: *On canonical forms of differential equations.*

In this paper the author introduces a class of congruences covariantly related to the surface and conjugate to the surface, each congruence being suitable as a substitute for Fubini's projective normal congruence. Each member of the class is determined by a particular choice of an invariant R with certain properties. As an application he gives a congruence of the class mentioned above and associated with a scroll surface. (Received March 27, 1930.)

267. Professor Harry Levy: *A theorem on linear connections.*

Given, in the space characterized by the paths

$$d^2x^i/ds^2 + \Gamma^i_{\alpha\beta}(dx^\alpha/ds)(dx^\beta/ds) = 0,$$

an arbitrary k-dimensional spread V_k, there exist infinitely many coordinate systems such that the paths through the points of V_k in directions linearly dependent on any $n-k$ preassigned directions, none of which are tangent to V_k, are given by linear equations. The author finds as a special case that seems to be of great interest that in a Riemannian space there always exists a coordinate system such that the geodesics through any point of a preassigned manifold in any direction orthogonal to that manifold are given by linear equations. Applications of this theorem to simplify well known results as well as to obtain new ones are given. (Received March 28, 1930.)

268. Professor Harry Levy: *On curves of zero curvatures in Riemann spaces.*

The author shows that if the kth curvature of a curve in a Riemann space is zero, the curve must lie on a k-dimensional spread generated by the ∞^k geodesics through a point in a direction linearly dependent on k directions, and conversely any curve contained in such a manifold has curvatures $1/\rho_m$, $m=1, 2, 3, \cdots, n-1$, which are zero for $m=k, k+1, \cdots, n-1$. It is further shown that the Frenet equations are valid for such a curve. (Received March 28, 1930.)

269. Miss E. T. Stafford: *On matrices conjugate with respect to the minimum equation.*

Matrices conjugate to a given matrix M with respect to the characteristic equation of M have been defined by Taber and Bennett for special cases and by Franklin in general. The conjugates defined by Franklin (Annals of Mathematics, vol. 23 (1921), p. 97) possess, for the characteristic equation, properties (1) and (2) here given for conjugates with respect to the minimum equation, but do not satisfy (3). The conjugates with respect to the minimum equation, as defined, have these properties: (1) conjugates are commutative as to multiplication, (2) the elementary symmetric functions of M and its conjugates coincide with the elementary symmetric functions of the roots of the minimum equation, (3) each conjugate is expressible as a polynomial in M. As a result of (1) and (2), each conjugate satisfies the minimum equation. In case the minimum equation is $(x-r)^p=0$, the conjugates form the unique set of linear polynomials in M which satisfy (1) and (2). The general conjugates are formed in an elementary manner, from conjugates of such matrices. Necessary and sufficient conditions that M and its conjugates form a basis for a linear algebra are given. (Received March 28, 1930.)

270. Mr. E. J. Finan: *A determination of the integral domains of the complete rational matric algebra of order* 4.

The domains considered are of the kind defined by Speiser which contain the principal unit. By means of two transformations which leave the discriminant invariant, any domain may be put into one of the canonical forms of which there are only a finite number for a given discriminant. All possible domains may be obtained from those in canonical form by transformations under which the set of all rational integral domains is closed. These results are an extension of those obtained by Du Pasquier. A determination of the domains for relatively small values of the discriminant is included. (Received March 18, 1930.)

271. Mr. E. B. Escott: *Amicable numbers.*

Two numbers are called amicable if each equals the sum of the aliquot divisors of the other, i.e., the sum of the divisors of the number, excluding the number itself. This can be expressed by means of the two equations $S(m)=m+n=S(n)$, where $S(m)$ is the sum of the divisors of m including 1 and m itself. Since the problem involves the question of the primality of the factors of the numbers considered, the solution requires many trials. This paper gives an original method by which the number of necessary trials is reduced very materially. The following is a list of the number of amicable pairs known at present: Iamblichus 1 (283–330 A.D.), Fermat 1 (1636), Descartes 1 (1638), Euler 59 (1750), Legendre 1 (1830), Paganini 1 (1866), Seelhoff 2 (1884), Dickson 2 (1911), Mason 14 (1921), Escott 31 (1930). (Received March 25, 1930.)

272. Dr. W. D. Baten: *The probable error of certain functions of the errors.*

This paper presents with proof and applications several theorems pertaining to the probable error of certain functions of the errors made in measurements. It is assumed that a is the true value of the quantity to be measured, m the measurement or observation. If the measurements are multiplied by a constant b, the probable value of the functions of the error of bm will differ from the probable value of these same functions of the error of m. This paper shows that there are values of b such that the probable value of certain functions of the error of bm is less than the probable value of these functions of the error of m. The probable value of certain functions of the error of the mean is also treated. General frequency laws for the error are used which include the discrete, the continuous, and a combination of the discrete and the continuous cases. Stieltjes integrals are used to prove most of the theorems. Special cases are treated in the corollaries. (Received March 27, 1930.)

273. Mr. G. C. Munro: *Systems of partial differential equations with constant coefficients.*

This paper deals with a system of linear homogeneous partial differential equations in n dependent variables, with constant coefficients, namely, the system

$$\frac{\partial x_i}{\partial u} = \sum_{j=1}^{n} a_{ij}x_j, \qquad \frac{\partial x_i}{\partial v} = \sum_{j=1}^{n} b_{ij}x_j, \qquad (i=1, \cdots, n),$$

where the a_{ij} and b_{ij} are constants. All possible conditions under which common solutions for the system may exist are discussed, and in each case the solutions are obtained explicitly. It is shown that all common solutions are exponential in form, that is, they consist of exponentials whose coefficients are polynomials in the independent variables. It is further shown that there exist fundamental sets of common solutions for the system if and only if multiplication between the matrices of the coefficients in the two sets of equations is commutative. If these matrices, A and B, are not commutative, the number of common solutions for the system is at most equal to the number of dependent variables, n, minus the rank of the matrix $AB-BA$. (Received March 15, 1930)

274. Dr. M. G. Boyce: *An envelope theorem and necessary conditions for a problem of Mayer with one variable end point.*

Let y denote the set $y_1, \cdots, y_n$. The problem studied is that of finding among the arcs $y=y(x)$ which satisfy a system of differential equations $\phi_\alpha(x, y, y')=0$, $\alpha=1, \cdots, m<n$, one which has a fixed initial point (x_1, y_1) and whose end point $[x_2, y(x_2)]$ minimizes the first of a set of functions $f_\sigma[x_2, y(x_2)]$, $\sigma=0, 1, \cdots, r<n$, while making the rest vanish. Multiplier

rules with transversality conditions have been derived for more general problems by Bolza and Bliss. In this paper further necessary conditions are given analogous to those of Weierstrass and Legendre. An envelope theory is then developed and is shown to lead to a Jacobi condition. Attention is given to the distinction between normal and abnormal admissible arcs, and it is found possible to make the proofs with weaker normality hypotheses than are customarily used in the Lagrange problems. (Received March 25, 1930.)

275. Professor L. E. Dickson: *Researches in the theory of numbers.*

New results on quadratic forms and diophantine equations in three or more variables, and on Waring's problem and its generalizations are given. The former appear in the author's current book, *Studies in the Theory of Numbers* (University of Chicago Press). (Received April 16, 1930.)

276. Mr. A. E. Ross: *On certain universal and zero indefinite ternary quadratic forms.*

In this paper we consider properly primitive indefinite ternary quadratic forms f with the invariants $\Omega = \pm 1$ and $\Delta = \mp D$, where $D > 0$ and $\not\equiv 0 \pmod 4$. We show that such a form f is universal if and only if $(F/\rho) = (-\Omega/\rho)$ for every odd prime factor ρ of D, and that every universal form f is a zero form. Here F is the reciprocal of f. The conditions $\Omega = \pm 1$ and $\Delta \not\equiv 0 \pmod 4$ are necessary in order that the form f be universal. This last fact was communicated to the writer by Mr. A. Oppenheim since the paper was written. Therefore the above theorem is of interest as it furnishes an easily applicable criterion for universality of any properly primitive indefinite ternary quadratic form. Write $D = PQ^2$, where Q is the largest square divisor of D, $P = 2^\alpha p_1 \cdots p_\pi$, where $p_1, \cdots, p_\pi$ are positive odd primes. Let σ be the number of distinct odd primes dividing Q but not P. We show that a form f is a zero form if and only if $(F/p_j) = (-\Omega/p_j)$ $(j = 1, \cdots, \pi)$, that there are 2^σ classes of zero forms f, and that the absolute value of the minimum of every form f which is not a zero form is unity. (Received April 18, 1930.)

277. Mr. A. E. Ross: *On universal indefinite quaternary quadratic forms* $\phi = f(x, y, z) + \alpha w^2$.

C. G. Latimer (Annals of Mathematics, (2), vol. 28 (1926–7), p. 327) had shown that the form $x^2 + y^2 - \alpha z^2 - \alpha w^2$ represents all integers if $\alpha \leqq 163$ is of the form $4k + 3$ and is the product of distinct positive odd primes. The same was shown by R. H. Marquis (Dissertation, University of Chicago, 1929) for $\alpha < 81 \equiv 1 \pmod 4$ and containing no square factors. Employing results of a former paper (abstract No. 36-1-38) we here prove the following general theorem which includes the results above as a very special sub-case. Let $f(x, y, z)$ be an indefinite ternary quadratic form whose determinant D is the product of distinct odd primes. Let F be the reciprocal of f. Let α be an integer $\not\equiv 0 \pmod 8$ and not divisible by the square of any prime factor p of D for which $(F/p) = -(-\Omega/p)$. Then the form $\phi = f(x, y, z) - \alpha w^2$ represents all integers. Here Ω is the first invariant of f. (Received April 18, 1930.)

278.* Professor E. T. Bell: *Residues of certain binomial coefficients for composite moduli.*

If $a, b, \cdots, c$ are integers greater than zero, and $p, q, \cdots, r$ are distinct positive primes, there is a simple explicit formula for the residue modulo $p^a q^b \cdots r^c$ of the coefficient of x^h in the expansion of $(1+x)$ to the power $ap+bq+\cdots+cr$. The formula holds also when some but not all of $a, b, \cdots, c$ are zero. (Received May 13, 1930.)

279. Professor E. T. Bell: *Unique decomposition.*

A complete theory is developed around the following problem, of which the theory provides a denumerable infinity of distinct solutions. A set closed under a single operation, called composition, is called an *ovum* if with respect to the operation the elements of the set satisfy the associative and commutative laws. Suppose that in an ovum there is a denumerable infinity of elements, or a finite number of elements, such that any element in this set is uniquely composable from elements of a subset of the set, and no element in the subset is composable out of elements of the subset. This obviously is the situation of unique factorization into primes in its lowest terms. From the data alone no more arithmetic is possible since, if composition be identified with the commutative and associative properties of multiplication or of addition, it is automatically identified with those of both. A complete arithmetic is both additive and multiplicative. The problem is, from the data as above to define with respect to the elements and their law of unique composition new elements which shall have a complete arithmetic, order relations included, such that any relation in the complete theory has a unique interpretation in terms of the data. In giving the solutions it became necessary to devise a theory of arithmetic divisibility for divisors of zero. The theory contains many new (but simple) concepts. In particular it provides an effective definition for the addition of ideals which is given in detail as an illustration in the next paper. (Received May 13, 1930.)

280. Professor E. T. Bell: *Rings whose elements are ideals.*

Given only the unique decomposition of an integer, considered as a principal ideal, in any algebraic number field, it is required (1) to order the ideals of the field; (2) from (1) and the original data to define addition, multiplication and subtraction of ideals so that, under these operations, which have abstractly all the properties of the similarly named operations in a ring, the ideals of the field shall form a ring; (3) to interpret the solutions of (2) in terms of the original data. This problem is solved as an example of the preceding theory. Addition in the solution is multiplication in the original field; multiplicaton is a new process which has close affinities with the fundamental properties of the G.C.D. and L.C.M. as defined for Dedekind ideals. In this, as in the general theory, it was necessary to discuss arithmetical divisibility for divisors of zero. The solution gives automatically the properties of "greater" and

* The papers beyond this point are to be read at meetings of which the reports have not yet been published.

"less," abstractly identical with the like for real numbers, with reference to algebraic numbers and ideals. This does not depend upon the norm, but upon the rational integers occurring as coefficients in the basis representation of integers or ideals. This again is but an instance of the theory of inequality for matrices whose elements are real numbers. (Received May 13, 1930.)

281. Professor E. T. Bell: *A theory of certain fields of points.*

Another type of solution of the problem of unique decomposition is presented. This is geometrical in character and refers to fields (the rational operations being given suitable geometrical interpretations) of points lying on a one dimensional variety in a space of n dimensions. The points on a sphere in n-space are represented parametrically by a well known transformation due to Green (Cambridge Philosophical Transactions, 1835). The set of all points on the sphere whose parameters are the coordinates of points on a sphere in the next lower space is said to define a set of cospherical points on the original sphere. The process is continued until a circle is reached. The points of the circle are then considered as the elements in a field after being placed in one to one correspondence, by a certain transcendental transformation, with all the points on the real axis. The theory concerning the circle or the points in the closed interval from 0 to 1 was given in a previous paper, *The real unit segment as a number field* (Abstract No. 36-5-218). The process is now reversed and fields of points in n-space are generated. Algebraically, all the operations and theorems can be interpreted as relations between ordered sets of n real numbers. Order relations for such ordered sets treated as elements are essential to the theory; they are abstractly identical with "greater" and "less" as referred to real numbers. (Received May 13, 1930.)

282. Dr. Arthur B. Brown: *An extension of the Alexander duality theorem.*

Let S be an n-manifold with the Betti numbers of an n-sphere. Let D be a proper sub-set of the points of S, possessing finite or infinite topological Betti numbers; and let K be a closed point set on a part of D which is open with respect to S. Then the following relations hold, where $R_{n-i-1}(K)$ denotes the Vietoris Betti number, and the other terms denote topological Betti numbers:

$$R_i(D - K) = R_i(D) + R_{n-i-1}(K), \quad i = 0, 1, \cdots, n-1.$$

This theorem is proved by use of the Alexander duality theorem (as extended by Alexandroff, Lefschetz and Alexander) and a theorem about overlapping complexes. Another proof has been found which is longer, but depends only on the Alexander theorem. (Received May 13, 1930.)

283. Professor Daniel Buchanan: *Periodic orbits in the problem of three bodies with repulsive and attractive forces.*

Two repellant particles under the attraction of a central nucleus will move in circular orbits the planes of which are parallel. The line joining the centres of these circles is normal to their planes and is bisected by the nucleus. The paper deals with periodic oscillations near these circular orbits. The configuration of the three bodies is similar to that of the helium atom. (Received May 13, 1930.)

284. Professor Daniel Buchanan: *Periodic orbits in the problem of four bodies with repulsive and attractive forces.*

The problem considered deals with the motion of two repellant particles which move subject to the attraction of two nuclei of equal mass. Periodic orbits for the particles are obtained which lie in the right bisector plane of the join of the nuclei. The configuration of the four bodies is similar to that of the normal hydrogen molecule. (Received May 13, 1930.)

285. Professor Florian Cajori: *The word "logarithm" used before the time of Napier.*

"Logarithm" is used in the compound word "logarithmanteia" by Casper Peucer in his *Commentarius* on methods of divination (Wittenberg, 1553). This fact I owe to Roscoe Lamont of Washington, D. C. Peucer associates the letters of the alphabet with triangular numbers. Divination is achieved by passing from numbers to letters or conversely. "Logarithmanteia" is derived from "logos" (word), "arithmos" (number) and "manteia" (divination). With Napier "logos" means ratio. The Englishman John Gaule in 1652 used "logarithmancy" in Peucer's sense of divination "by logarithmes." Peucer is mentioned by Robert Burton in 1621. It is possible that Napier got his word "logarithm" (1614) from Peucer. (Received May 13, 1930.)

286. Professor A. F. Carpenter: *Note on ruled surfaces and their developables.*

Each plane π fixed in position with respect to a ruled surface R determines with R two developable surfaces. One cuts π in the curve of intersection of π and R, the other cuts π in a curve whose points are those which correspond to π in the null-systems of the osculating linear complexes of R, and they intersect each other in a curve whose points are the poles of π with respect to the quadrics which osculate R along its line elements.

The proof makes use of conditions on the coordinates of π developed by the author in a previous paper (Annali di Matematica, (3), vol. 27 (1917), p. 285). (Received May 13, 1930.)

287. Ronald M. Foster: *Mutual impedances of grounded circuits.*

The mutual impedance between circuits consisting of insulated wires of negligible diameter lying on the surface of the earth and grounded at their terminals is

$$
\begin{aligned}
Z_{12} &= \frac{1}{2\pi\lambda}\int\int\left\{\frac{d^2}{dSds}\left(\frac{1}{r}\right) + \frac{\cos\epsilon}{r^3}[1 - (1+\gamma r)e^{-\gamma r}]\right\}dSds \\
&= \frac{1}{2\pi\lambda}\left(\frac{1}{Aa} - \frac{1}{Ab} - \frac{1}{Ba} + \frac{1}{Bb}\right) + i\omega N_{Ss} \\
&\quad + (1-i)\frac{1}{3}(8\pi\lambda\omega^3)^{1/2}AB\cdot ab\cdot\cos\theta + \cdots \\
&= \int\left[\frac{1}{\pi\lambda X^2} - \frac{\gamma}{\pi\lambda X}K_1(\gamma x)\right]\cos\epsilon ds + \cdots.
\end{aligned}
$$

The earth is assumed flat, semi-infinite in extent, and of uniform conductivity λ. All displacement currents are neglected,—the frequency $\omega/(2\pi)$ thus being limited to relatively low values. The propagation constant is $\gamma=(i4\pi\lambda\omega)^{1/2}$. In the first form the integrations are extended over the two wires S and s, their elements dS and ds being separated by the distance r and making the angle ϵ with each other. The second form gives the first three terms in the expansion for low frequencies; N_{Ss} is the mutual Neumann integral between the two wires S and s, which extend from A to B and from a to b, respectively, and θ is the angle between the straight lines AB and ab. The third form gives the first term in the expansion for a long straight wire S and any wire s located near the mid-point of S, x and ϵ being respectively the distance from the element ds to S and the angle between ds and S. (Received May 13, 1930.)

288. Professor Raymond Garver: *An application of a theorem of Junker.*

Considerable use has been made by Hermite, Cayley, Brioschi and others of transformations on algebraic equations which lead to transformed equations whose coefficients have certain invariantive properties. However, one very general transformation of this type, due to Junker, has apparently escaped attention almost entirely. The present paper mentions several problems in which it may be of value, and considers in detail its use in the reduction of the general quartic to binomial form. The method employed seems to be of some interest apart from its connection with Junker's theorem; in fact, the main result of this paper can be obtained quite conveniently without reference to the latter. (Received May 13, 1930.)

289. E. R. Hedrick: *Varieties of polar singularities of non-analytic functions.*

In this paper the author shows the possibilities of different varieties of singularities of non-analytic functions which have some of the properties usually associated with poles of analytic functions. For one type, called analytic poles, the author shows that the classical theorems regarding functions which have only poles remain valid for non-analytic functions. (Received May 13, 1930.)