ON THE DIRECT PRODUCT OF A DIVISION AND A TOTAL MATRIC ALGEBRA*

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This paper establishes certain theorems concerning an algebra A which is expressible as the direct product† of a division algebra D and a total matric algebra M. It is moreover not assumed that D and M are subalgebras of A. We let δ and n^2 represent the orders of D and M respectively. It follows that δn^2 is the order of A. We represent the modulus of A by be where b and e are the respective moduli of D and M. In agreement with the usual notation, we write

$$e = \sum e_{ii}, \ (i = 1, \cdots, n),$$

where e_{ij}, $(i,j = 1, \cdots, n)$, are the basal units of M.

For the proof of Theorem 1, we express the zero elements of algebras A, D and M by Z, z_d and z_m respectively. Thereafter we employ the symbol 0 without ambiguity. Since the elements of D and M are commutative with each other and a zero element of an algebra is unique, we have‡ $Z = z_d z_m$.

THEOREM 1. *If $dm = Z$, where d and m are any elements of D and M, respectively, then either $d = z_d$ or $m = z_m$.*

For, if $d \neq z_d$, it possesses an inverse d^{-1}. It follows that

$$bm = d^{-1}Z = d^{-1}z_d z_m = z_d z_m = Z.$$

Writing

$$m = \sum_{i,j=1}^{n} \alpha_{ij} e_{ij},$$

we have

$$\sum_{i,j=1}^{n} \alpha_{ij} b e_{ij} = Z.$$

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† Dickson, *Algebras and their Arithmetics*, p. 72.

‡ In the proof, let $Z = z_1 z_2$, where z_1 is in D and z_2 in M. Then

$$Z = Z \cdot z_d z_m = z_1 z_2 \cdot z_d z_m = z_1 z_d \cdot z_2 z_m = z_d z_m.$$

If $m \neq z_m$, there must be some $\alpha_{rs} \neq 0$. Multiplying the equation above on the left by e_{rr} and on the right by e_{ss}, we obtain $\alpha_{rs} b e_{rs} = Z$, whence $b e_{rs} = Z$. Multiplying on the left and right by e_{ir} and e_{si} respectively and summing with respect to i, we obtain

$$b \sum e_{ii} = be = Z,$$

where be is the modulus of A. It follows that $m = z_m$ in case $d \neq z_d$.

THEOREM 2. *If $d_1 m_1 = d_2 m_2$, where d_1 and d_2 are non-zero elements of D, and m_1 and m_2 are non-zero elements of M, then $d_1 = d_2/\alpha$ and $m_1 = \alpha m_2$, where α is a scalar.*

For, let

$$m_1 = \sum_{i,j=1}^{n} \alpha_{ij} e_{ij}, \quad m_2 = \sum_{i,j=1}^{n} \alpha'_{ij} e_{ij}.$$

Then by hypothesis, we have

$$\text{(i)} \qquad \sum_{i,j=1}^{n} (\alpha_{ij} d_1 - \alpha'_{ij} d_2) e_{ij} = 0.$$

Since $m_1 \neq 0$, there is some $\alpha_{rs} \neq 0$. We multiply (i) on the left and right by e_{rr} and e_{ss} respectively, obtaining

$$(\alpha_{rs} d_1 - \alpha'_{rs} d_2) e_{rs} = 0.$$

From Theorem 1, since $e_{rs} \neq 0$, we have

$$d_1 = \frac{\alpha'_{rs}}{\alpha_{rs}} d_2,$$

where $\alpha'_{rs} \neq 0$ since $d_1 \neq 0$. Accordingly $d_1 = d_2/\alpha$, where $\alpha = \alpha_{rs}/\alpha'_{rs}$. Hence $d_1 m_1 = d_2 m_2$ implies $d_1(m_1 - \alpha m_2) = 0$, whence $m_1 = \alpha m_2$ by Theorem 1.

THEOREM 3. *If dm is idempotent in A, then $d = b/\alpha$ and $m = \alpha m'$, where α is a scalar and m' is idempotent in M.*

For, from $d^2 m^2 = dm$, we have $dm^2 = bm$, on multiplying by d^{-1}. From Theorem 2, $d = b/\alpha$ and $m^2 = \alpha m$. Now let $m = \alpha m'$; then

$$m'^2 = \frac{m^2}{\alpha^2} = \frac{\alpha m}{\alpha^2} = \frac{m}{\alpha} = m'.$$

This proves the theorem.

THEOREM 4. *If e' and e'' are any elements of M, then the algebra $e'Ae''$ is the direct product $D\times e'Me''$.*

Since $A=D\times M$, then $e'Ae''=D(e'Me'')$, for e' and e'' are commutative with elements of D.

Let n' and n'' represent the orders of $D(e'Me'')$ and $e'Me''$ respectively. Then* $n'\leqq \delta n''$, where δ is the order of D. In case $e'Me''<M$, we can think of the basal units of M as made up of n'' linearly independent elements of the algebra $e'Me''$ together with certain other n^2-n'' elements in M. Hence the products of these n'' elements of $e'Me''$ by the basal units of D give $\delta n''$ linearly independent elements of $e'Ae''$, since they may be considered as certain of the δn^2 basal units of $A=D\times M$. It follows that $n'=\delta n''$.

DEFINITION. A set of primitive idempotent elements is said to be supplementary in case their sum equals the modulus and if further the product of each pair in either order is zero.

THEOREM 5. *Each algebra e_iMe_j is of order* 1, *where e_i and e_j belong to a supplementary set of primitive idempotent elements.*

For, since every total matric algebra is simple,† it follows that we may apply the method of §51 of Dickson's *Algebras and their Arithmetics* with M replacing the algebra A. The modulus $\sum e_{ii}$ of M may be written in the form

$$\sum e_{ii}=\sum_{k=1}^{m} e_k,$$

where the $e_1, \cdots, e_m$ are a set of supplementary primitive idempotent elements which include e_i and e_j. This follows from Theorem 3, p. 57 of Dickson's *Algebras and their Arithmetics*, and the last few lines on p. 49 of the same text. We obtain

$$M=\sum_{i,j=1}^{m} M_{ij},$$

where $M_{ij}=e_iMe_j$ and where each of the m^2 algebras M_{ij} is of the same order t. Finally, we are able to write $M=D\times M'$, where M' is a total matric algebra of order m^2. On the other hand we may write $M=(1)\times M$. Since the expression of a simple algebra as the direct product of a division and a total matric algebra is unique‡ in the sense of equivalence, it follows

* Dickson, *Algebras and their Arithmetics*, §18.

† Dickson, *Algebras and their Arithmetics*, p. 80.

‡ Scorza, *Corpi Numerici e Algebre*, 1921, pp. 346–352.

that $M \cong M'$, whence $m=n$. But M was of order tm^2. Accordingly $t=1$ is the order of each of the algebras $M_{ij}=e_iMe_j$.

COROLLARY. *A supplementary set of primitive idempotent elements of a total matric algebra of order n^2 contains exactly n elements.*

THEOREM 6. *If e' is a primitive idempotent element of M, then be' is a primitive idempotent element of $A=D\times M$, and conversely.*

For, consider $e'Ae'$. From Theorem 4, we have $e'Ae' =D\times e'Me'$, where $e'Me'$ may be considered as a total matric algebra of order 1, whose modulus is e'. We must show that be' is the only idempotent of $e'Ae'=be'\cdot A\cdot be'$. We note that any element of $e'Ae'$ may be written in the form $de'me'$, where $d\leqq D$ and $m\leqq M$. If this element is idempotent, then from Theorem 3, replacing A by $e'Ae'$ and M by $e'Me'$, we have $d=b/\alpha$ and $e'me'=\alpha m'$, where α is a scalar and m' is idempotent in $e'Me'$. But $e'Me'$ contains no idempotent other than e'. It follows that $de'me'=be'$ is the only idempotent in $e'Ae'$.

Conversely, let be' be a primitive idempotent element of A and suppose m is any idempotent of $e'Me'$. Then bm is idempotent in

$$e'Ae'=D\times e'Me'.$$

Hence $bm=be'$, since be' is primitive in A, and hence

$$be'\cdot A\cdot be'=e'Ae'$$

has the single idempotent be'. From Theorem 1, it follows that $m=e'$. Accordingly e' is a primitive idempotent of M.

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