at home with new ideas, and when they are wisely chosen they can exhibit the variety that exists in a subject and suggest new lines of thought. Even simpler problems are not to be despised, for it is not a base thing to get pleasure from manipulating a problem into a practicable answer. One can never be sure that the most uncompromising devotee of existence theorems does not some times seek amusement in such indulgence.

All persons who have given any attention at all to difference equations must hope that the subject will find an appropriate place in the mathematical structure. It would seem that extensive treatises on analysis should devote some space to it, at least to the extent of treating the gamma function in a way different from the traditional one, and going a little beyond the gamma function. It is to be hoped that the material in Batchelder's book may help to make such treatments possible as well as stimulate further study of the subject.

K. P. WILLIAMS

HELLINGER-TOEPLITZ ON INTEGRAL EQUATIONS

Integralgleichungen und Gleichungen mit unendlichvielen Unbekannten. By E. Hellinger and O. Toeplitz. Reprint from the Encyclopädie der Mathematischen Wissenschaften with the addition of a preface by E. Hilb and of a special subject-index. Leipzig-Berlin, Teubner, 1928. Pp. 1335–1616.

In his preface Hilb points out that the book under review appears after a quarter of a century of research in the field of the theory of integral equations. It must be considered, therefore, as a survey of results obtained, and as an account of problems which remain still unsolved. "During several years of cooperative work the authors scrutinized the whole literature as to methods, results and their comparative range." We agree with Hilb that this "Report is indispensable to anybody who desires to penetrate deeply into this subject so extraordinarily important for its applications." An attentive reader, even well versed in the subject, will find many novel features in treating old and new questions, features which are extremely illuminating and inspiring; he will welcome the successful efforts of the authors to unify the multitude of existing methods and to present these methods as parts of a harmonious whole. As notable instances of this kind we may mention the treatment of completely continuous forms (pp. 1403–1413); of normal matrices (p. 1562; this seems to be a new notion introduced by the authors and proving to be quite useful in a number of recent investigations); of symmetrizable kernels (pp. 1536–1543) and matrices (pp. 1563–1575); of a general principle which could be designated as a "principle of preservation of the type" of kernels or matrices (pp. 1391–1392, 1431–1433). It is undoubtedly a good idea (although a departure from the usual style of the Encyklopädie) to give *proofs* of some facts of fundamental importance; the reader will be also pleasantly surprised to find references to some *facts not previously published* (Toeplitz, p. 1573; Toeplitz-Schmidt, p. 1575; Szász, p. 1522). It is hardly necessary to mention that the bibliography of the Report is extremely rich and shows that the authors have canvassed almost everything hitherto published on the theory of integral equations.

All this being granted it seems desirable now to call attention to some points in the Report, which may present a certain interest.

1. The statement of any theorem consists of two parts of equal importance: (a) hypotheses, (b) conclusions. The balance between these two parts is not always preserved in the Report: references to some theorems are lacking in the exact formulation of the conditions of their validity and are, in this sense, somewhat misleading. As examples we may quote the references to St. Bóbr (p. 1446) who treated not the general case $p>1$ mentioned in the text, but only the particular case $p\geqq 2$; Mollerup (p. 1375); Lalesco (p. 1550) who proved the theorem of the text under restrictive conditions, while the proof of the general theorem has never been published, so far as we are aware; it is not clear what is meant on p. 1376 (top) by the statement that the resolvent and pseudo-resolvent are meromorphic functions of λ, the notion of the pseudo-resolvent being defined only for a discrete set of values of λ (characteristic values); there is an apparent contradiction in the references to General Analysis, on pp. 1475–1476: while the references on p. 1475 seem to indicate that there are results in General Analysis concerning and unifying the theory of bounded forms, the next page implies that Moore's hypotheses in Hilbert's case lead only to a case which is even more special than completely continuous forms.

2. The reader will find a more interesting example in this same subject in the treatment of general spaces on pp. 1446–1448. The authors raise here a very important question as to the possibility of extending Hilbert's theory of completely continuous forms to general spaces. Following an idea of Helly* they consider a space $(x)=(x_1,x_2,x_3, \cdots)$ of infinitely many dimensions, on a sub-space of which there is defined a "distance-function" $D(x_1,x_2, \cdots)$, satisfying the following postulates:

(i) $D(\lambda x_1, \lambda x_2, \cdots) = |\lambda| D(x_1, x_2, \cdots)$,

(ii) $D(x_1+y_1, x_2+y_2, \cdots) \leqq D(x_1, x_2, \cdots) + D(y_1, y_2, \cdots)$,

(iii) $D(x_1, x_2, \cdots) \geqq 0$; $D(x_1, x_2, \cdots) = 0$ implies $x_1 = x_2 = \cdots = 0$.

To these postulates they add a new one:

(iv) $$D(x_1, x_2, \cdots) = D(|x_1|, |x_2|, \cdots).$$

Simultaneously with the space (x) the authors consider the "polar" space (u), whose distance-function $\Delta(u_1, u_2, \cdots)$ is defined by the condition

$$\Delta(u_1, u_2, \cdots) = \text{upper limit} \left|\sum_{n=1}^{\infty} u_n x_n\right|, \text{ if } D(x_1, x_2, \cdots) \leqq 1,$$

provided the upper limit in question is finite. On page 1448 (footnote 224) it is stated that under the conditions indicated above the "Auswahlverfahren" and "Abspaltungsverfahren" (which calls for the use of Hilbert's theory of completely continuous forms) can be extended to the spaces (x), (u). As

* *Über Systeme linearer Gleichungen mit unendlichvielen Unbekannten*, Monatshefte für Mathematik und Physik, vol. 31 (1921), pp. 60–91.

interesting as this observation may be, the following examples show that some additional postulates are necessary in order to make it correct.

Ex. 1. The "points" are all the continuous functions $x(t)$ defined on (0,1). The coordinates of the point x are the values assumed by $x(t)$ at all the rational points of (0,1). The distance-function $D(x_1, x_2, \cdots)$ is defined by

$$D(x_1, x_2, \cdots) = \int_0^1 | x(t) | dt.$$

It is now readily seen that, although the postulates (i-iv) are satisfied, there can be no question here of any "Auswahlverfahren" and, furthermore, the polar space (u) does not exist but reduces to the single point $(0,0, \cdots)$*.

Ex. 2. The space (x) consists of all the sequences $\{x_\nu\}$ for which $\sum_{\nu=1}^{\infty} |x_\nu|$ converges. The space (u) consists of all the sequences $\{u_\nu\}$ for which upper limit $|u_\nu|$ is finite. If the distance-function $D(x_1, x_2, \cdots)$ is defined by

$$D(x_1, x_2, \cdots) = \sum_{\nu=1}^{\infty} |x_\nu|,$$

it is readily seen† that the distance-function $\Delta(u_1, u_2, \cdots)$ is simply upper limit $|u_\nu|$. The postulates (i-iv) are satisfied again, but the form $\sum u_\nu x_\nu$ is not completely continuous (in the sense of p. 1401).

3. Although the bibliographical references of the Report as a rule are extremely complete, there are a few gaps, of which some, in the reviewer's opinion, are serious. Considerably more space (than half a page) should be given to the account of the fundamental work of Carleman, *Sur les équations intégrales singulières*, etc., which is the most important contribution to the theory of singular integral equations since the publications of Hilbert, Weyl, Hellinger, and Toeplitz. The significance of Carleman's work lies not only in the results obtained but also in the method used. Carleman, in distinction from Hellinger and Toeplitz, obtains his results without using the theory of forms in infinitely many variables; this procedure is particularly convenient in the general theory of non-bounded operators (compare the recent investigations of J. von Neumann and M. H. Stone). Carleman's thesis (*Über das Neumann-Poincarésche Problem für ein Gebiet mit Ecken*, Uppsala, 1916, iv+195 pp.) is not mentioned at all in the Report. It is true that the title of the thesis indicates its potential-theoretical tendency which is outside the scope of the Report. But it is also true that this thesis contains many results of importance in the general theory of integral equations, for instance a very interesting application of "Abspaltungsverfahren" to the case of an integrable kernel (under conditions more general than those used by Dixon, p. 1388, footnote 95); a treatment of symmetrizable kernels where many of the results of Mercer (p. 1543) are obtained a few years earlier than by Mercer; etc. The authors do not mention an important result due to Carleman (Acta Matematica, vol.

* This example was suggested to the reviewer by N. Wiener. Of course this example would be ruled out if the authors had formulated the property of *completeness* of the space (x).

† See Hahn, *Über Folgen linearer Operationen*, Monatshefte, vol. 32 (1922), pp. 1–88.

41(1918), pp. 377–384) that 2 is the exact value of the exponent of convergence of the sequence of the characteristic values of a nonrestricted continuous kernel. A wholly erroneous statement concerning the work of Carleman is to be found in the Report. On p. 1531, Footnote 453, it is stated that Carleman proved the *convergence* of the series $\sum(\lambda_\nu^+)^{1/(1-\alpha)}$ where λ_ν^+ are the positive characteristic values of the kernel $|s-t|^{-\alpha}H(s,t)$, $H(s,t)$ being continuous and $H(s,s)>0$ on a subinterval of (a,b), and $0<\alpha<1$. However, what Carleman actually proved is the *divergence* of the series $\sum(1/\lambda_\nu^+)^{1/(1-\alpha)}$! A. Pell's thesis should have been mentioned on p. 1495 before Lichtenstein's paper, in connection with certain integro-differential equations.

4. The attitude of the authors of the Report toward the methods of General Analysis is sufficiently clearly expressed on pp. 1471–1476 and 1595-1596. This attitude can be explained partly by the fact that, pending the publication of a comprehensive monograph devoted to this contribution of the American mathematical school, the most interesting results of General Analysis are inaccessible not only to foreign mathematicians but even to the majority of those in this country. Still we find statements like the one on p. 1596 too rash: since neither methods nor results existed for the general elementary divisors' theory of bilinear matrices, no new results could be obtained here by General Analysis (Für die Schule von E. H. Moore fehlte darum hier jeder Ansatspunkt).

5. Several passages scattered throughout the Report create an impression that the authors consider the facts (Lösungs*tatsachen*) as more important than methods of obtaining them (Lösungs*formeln*). This point of view is perfectly justified if it should mean that, of two methods leading to the same formal results, the preference should be given to the one which uses the minimum of assumptions. There are numerous cases, however, (and not only in applications) where the method used is the most important part of the problem. From this point of view it seems desirable that some particular problems (for instance, the moment problem, inversion of definite integrals, and reciprocal relations) should have been given more attention in the Report. An impartial reader may also find that the special method of *forms* in infinitely many variables occupies too much of the foreground of the picture in the Report, as compared with the theory of *integral equations* as such. It should be finally observed that applications are almost completely omitted from the scope of the Report, which was certainly the right thing to do; another article of the Encyklopädie devoted to the applications of the theory of integral equations is therefore an obvious necessity.

The defects mentioned above detract little from the great value and interest of the Report, which will for a long time occupy an important place on the desks of those who are interested in the growth of the immense structure of the theory of integral equations.

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