CLASSES OF DIOPHANTINE EQUATIONS WHOSE POSITIVE INTEGRAL SOLUTIONS ARE BOUNDED*

BY D. R. CURTISS

1. *Introduction.* If upper bounds have been found for the positive integral solutions of a diophantine equation, the problem of obtaining all such solutions is reduced to making a finite number of trials. It may therefore be of interest to note certain cases where upper bounds are given by simple algebraic processes. Hereafter the term *solution* will always mean a solution in positive integers.

Our starting point is the observation that if $P(t)$ is a polynomial in t, then all positive values of x that satisfy the inequality $P(1/x) \geqq 0$ are bounded if (and only if) the term of lowest degree in $P(t)$ has a negative coefficient.

2. *A Type whose Solutions are always Bounded.* Every algebraic diophantine equation in n variables $x_1, x_2, \cdots, x_n$ can be thrown into a form where the right side is zero and the left side is a polynomial in the reciprocals of the x's. When this has been done, the first type here to be considered is the following:

$$F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right) - k = 0, \tag{1}$$

where F is a polynomial all of whose coefficients are positive, while k is a positive constant, and $F(0, 0, \cdots, 0) = 0$.

The positive integral solutions of every equation of type (1) *are bounded.*

To prove this statement, and to show how to obtain bounds for the solutions, let us first consider a solution such that $x_1 \leqq x_2 \leqq \cdots \leqq x_n$. We shall then have

* Presented to the Society, December 28, 1927.

$$F\left(\frac{1}{x_1}, \frac{1}{x_1}, \cdots, \frac{1}{x_1}\right) - k \geqq 0, \tag{2}$$

since each term of F is not decreased when another x is replaced by x_1. The term of lowest degree in (2), $-k$, is negative, hence, from the observation made in the second paragraph of this paper, x_1 must be bounded. To obtain an explicit upper bound we note that, since $1/x_1^n \leqq 1/x_1$ when n and x_1 are positive integers, and since all the terms of F are positive, we have

$$F\left(\frac{1}{x_1}, \frac{1}{x_1}, \cdots, \frac{1}{x_1}\right) \leqq \frac{1}{x_1} F(1, 1, \cdots, 1).$$

Hence when this result is applied to (2) we obtain

$$\frac{1}{x_1} F(1, 1, \cdots, 1) - k \geqq 0,$$

or

$$x_1 \leqq \frac{1}{k} F(1, 1, \cdots, 1). \tag{3}$$

Usually a lower bound than this can be derived by finding a closer approximation for the (unique) positive root of the equation obtained from (2) by retaining only the sign of equality.

Let us now find a bound for $x_r (r \leqq n)$ when upper bounds X_j have been assigned for each x_j from $j=1$ to $j=r-1$ in a solution where the x's form an ascending sequence from x_1 to x_n. We write (1) in the form

$$\begin{aligned} &F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right) \\ &- F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, 0, 0, \cdots, 0\right) \\ &= k - F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, 0, 0, \cdots, 0\right). \end{aligned} \tag{4}$$

The first two lines of (4) reduce to a polynomial

$$F_r\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, \frac{1}{x_r}, \frac{1}{x_{r+1}}, \cdots, \frac{1}{x_n}\right)$$

each of whose terms is positive and involves at least one of the variables $x_r, \cdots, x_n$. Its value will not be decreased if each of the variables $x_{r+1}, \cdots, x_n$ is replaced by x_r. The last line of (4) is positive for all x's from x_1 to x_{r-1} that belong to solution systems, since the F function that appears here consists only of certain terms of the complete F in equation (1) and must lack some of the terms* of the complete F when $r-1<n$. Hence the inequality derived from (4),

$$F_r\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, \frac{1}{x_r}, \frac{1}{x_r}, \cdots, \frac{1}{x_r}\right)$$
$$-\left[k - F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, 0, 0, \cdots, 0\right)\right] \geqq 0,$$

is of the type $P(1/x_r) \geqq 0$, with term of lowest degree in $1/x_r$ negative. It follows that x_r is bounded for each set of values of the preceding x's. If these preceding x's are bounded, there must be an upper bound for all the values of x_r that belong to solution systems. Since we have shown that x_1 is bounded, it follows that all the x's are bounded, in solutions where $x_1 \leqq x_2 \leqq \cdots \leqq x_n$. We conclude at once that they are bounded for every order of relative magnitudes.

If the x's are arranged in order of magnitude from x_1 to x_n, an explicit upper bound for x_r, of which that given by (3) for x_1 is a special case, is indicated by the inequality

$$(5) \qquad x_r \leqq \frac{1}{m_r} F_r(1, 1, \cdots, 1) = X_r,$$

where m_r is the least positive value of

$$k - F(1/x_1, 1/x_2, \cdots, 1/x_{r-1}, 0, 0, \cdots, 0)$$

for $x_j \leqq X_j$, $(j=1, 2, \cdots, r-1)$. Since m_r may be difficult to

* We assume that each of the variables $x_1, \cdots, x_n$ is explicitly present in the F function of equation (1).

evaluate, we note that (5) can be replaced by the weaker inequality

$$(6) \qquad x_r \leqq D_r F_r(1,1,\cdots,1) = X_r',$$

where D_r is the product of all the denominators of terms of

$$k - F(1/X_1', 1/X_2', \cdots, 1/X_{r-1}', 0, 0, \cdots, 0).$$

As an illustration, consider the equation

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 1.$$

Here $F(1, 1, \cdots, 1) = n$, and $F_r(1, 1, \cdots, 1) = n-r+1$. The value of $1/m_r$ is u_r, where $u_1 = 1$, $u_{k+1} = u_k(u_k+1)$.* Thus from (5) we obtain the upper bounds

$$X_r = (n-r+1)u_r, \quad (r = 1,2, \cdots, n).$$

From (6) we have another set of bounds X_r', such that

$$X_r' = (n-r+1)X_1'X_2' \cdots X_{r-1}'.$$

Hence

$$X_1' = n, \ X_2' = (n-1)n,$$
$$X_r' = (n-r+1)(n-r+2)(n-r+3)^2(n-r+4)^4 \cdots n^{2^{r-2}}, r>1.$$

3. *More General Types.* Equations of more general type than (1) can be written in the form

$$(7) \qquad F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right) = G\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right),$$

where F and G are polynomials all of whose coefficients are positive; we suppose all possible cancellations to have been performed. One or more of the variables may not be present in F, and the same may be true of G, but no variable is to be absent from $F-G$. We now investigate bounds for solutions where $x_1 \leqq x_2 \leqq \cdots \leqq x_n$.

If there is a constant term on either side of (7), x_1 *is bounded.*

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* See *On Kellogg's diophantine problem,* American Mathematical Monthly, vol. 29 (1922), p. 380.

For, if, for example, we have $G(0, 0, \cdots, 0) = k \neq 0$, then $F(1/x_1, 1/x_1, \cdots, 1/x_1) \geqq k$, and formula (3) gives a bound for x_1. If (7) has no constant term, we consider the two inequalities derived from (7),

$$F\left(\frac{1}{x_1}, \frac{1}{x_1}, \cdots, \frac{1}{x_1}\right) - G\left(\frac{1}{x_1}, 0, \cdots, 0\right) \geqq 0,$$

$$G\left(\frac{1}{x_1}, \frac{1}{x_1}, \cdots, \frac{1}{x_1}\right) - F\left(\frac{1}{x_1}, 0, \cdots, 0\right) \geqq 0.$$

If the term of lowest degree in $1/x_1$ on the left of either of these inequalities is negative, we conclude that x_1 is bounded.

Again, using the notation of the earlier part of this paper, we deduce from (7) the inequality

$$(8) \qquad F_r\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, \frac{1}{x_r}, \frac{1}{x_r}, \cdots, \frac{1}{x_r}\right) - \left[G\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, 0, 0, \cdots, 0\right) - F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, 0, 0, \cdots, 0\right)\right] \geqq 0,$$

and another in which F_r is replaced by G_r, and F and G are interchanged. Unless the expression in brackets is zero, one of these inequalities is of the type $P(1/x_r) \geqq 0$, with negative constant term. Hence *unless the pair of equations*

$$(9) \qquad F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right) = G\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right),$$

$$F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, 0, 0, \cdots, 0\right) = G\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, 0, 0, \cdots, 0\right)$$

has a solution in positive integers $x_1, x_2, \cdots, x_n$, arranged in that order of magnitude, every solution of (7) *so ordered has x_r bounded if all the preceding x's are bounded.* From this we at once obtain a sufficient condition that all the x's be bounded.

If the above condition fails we may replace the expression in brackets in (8) by

$$G\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_r}, 0, 0, \cdots, 0\right)$$
$$- F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_r}, 0, 0, \cdots, 0\right),$$

and F_r by F_{r+1}. If in the new inequality thus obtained the term of lowest degree in $1/x_r$ is negative, or if it is negative in the companion inequality, we infer again that x_r is bounded if this is true of the preceding x's.

An example to show that the above conditions may not be fulfilled is given by the equation

$$1 + \frac{1}{x_2} = \frac{1}{x_1} + \frac{1}{x_3^2}.$$

Here for $r = 2$ the second equation of system (9) becomes

$$1 = \frac{1}{x_1},$$

and the pair of equations (9) is satisfied by $x_1 = 1$, $x_2 = x_3^2$, where x_2 and x_3 are not bounded. On the other hand, many equations whose solutions are bounded escape these tests; an example is

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_1} - \frac{3}{x_2} = 0,$$

whose only solution in positive integers is $x_1 = x_2 = 1$.

4. *Algebraic Equations with Positive Integral Roots.* A corollary of the first theorem of this paper concerns itself with algebraic equations

$$x^n - a_1 x^{n-1} + a_2 x^{n-2} - \cdots + (-1)^n a_n = 0,$$

all of whose roots are positive integers. There is, of course, an infinite number of such equations for any given n. But there is only a finite number whose coefficients satisfy a relation

$$F\left(\frac{a_1}{a_n}, \frac{a_2}{a_n}, \cdots, \frac{a_{n-1}}{a_n}\right) = k,$$

where F is a polynomial with positive coefficients and $k>0$; for F is a polynomial in the reciprocals of the roots, and, when thus expressed, F has no constant term, so that the first theorem of this paper applies. We could obtain upper bounds for the roots, and therefore for the a's, by the methods of this paper. For example, if $a_{n-1}=a_n$, and if $x_1, x_2, \cdots, x_n$ are the roots, the x's must be solutions of the equation

$$\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n} = 1,$$

which has been discussed in § 2.

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ERRORS IN KRAITCHIK'S TABLE OF LINEAR FORMS

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Tables of the linear forms that belong to a given quadratic residue D, or in other words, the linear divisors of the quadratic form t^2-Du^2 were first published by Legendre.* A list of errors in these fundamental tables has been given by D. N. Lehmer.† Kraitchik‡ has recalculated and extended these tables to the limit $D=\pm 250$. It is of great importance in using the table that every entry be correct. Therefore in constructing his factor stencils, D. N. Lehmer found it advisable to make a new table by means of a more or less graphical method.§ This table which has not been pub-

* *Théorie des Nombres*, 1st. ed., Tables III–VII, 1798.

† This Bulletin, vol. 8 (1902), p. 401. See also the correction in this Bulletin, vol. 31 (1925), p. 228.

‡ *Théorie des Nombres*, vol. 1, p. 164–186, Paris, 1922. *Recherches sur la Théorie des Nombres*, vol. 1, p. 205–215, Paris, 1924.

§ This Bulletin, vol. 31 (1925), pp. 497–498.