CONTINUOUS CURVES IN WHICH EVERY ARC MAY BE EXTENDED*

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1. *Introduction.* An arc α belonging to a point set K is said to be a *maximal arc* of K if there is no arc which belongs to K and contains the arc α as a proper subset. S. Mazurkiewicz‡ proved in 1921 that every arc of a bounded acyclic continuous curve§ M is a subset of a maximal arc of M. However there are many types of continuous curves in which every arc of a continuous curve is a subset of a maximal arc and which are not acyclic; and in a recent paper‖ we have given a necessary and sufficient condition in order that a continuous curve have this property. In this note we shall treat the opposite problem of finding under what conditions no arc is a subset of a maximal arc. We shall say that an arc α of a point set K is *extendible* in K if there exists an arc β of K of which α is a proper subset. If X and Y are the end points of α and Y is an interior point of β, then we say that α is *extendible in K in the direction XY*. And if there exists an arc β of K containing α and such that both end points of α are interior points of β, then we say α is *extendible in K in both directions*. In this note we shall give two necessary and suffi-

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cient conditions that every arc of a continuous curve M be extendible in M in both directions. The first of these conditions is that M shall not contain a certain type of point. From this it may be seen that in order that every arc of M be merely extendible in M it is necessary and sufficient that M contain at most one point of this type.

All of the point sets mentioned in this paper are assumed to lie in a two-dimensional euclidean space E_2.

2. *Preliminary Lemmas.* P. Urysohn[*] has shown that every point of a continuous curve may be ϵ-separated,[†] for any positive number ϵ, by a set consisting of a finite number of continua. By a result due to G. T. Whyburn and the author,[‡] each of these continua may be replaced by a continuous curve so that we still have an ϵ-separation. Hence in characterizing types of points of a continuous curve through their ϵ-separating sets we may consider only the sets of continuous curves which form an ϵ-separating set. In the plane we may show that we need consider only arcs[§] and simple closed curves.

LEMMA A. *If P is a point of a continuous curve M and ϵ is a positive number, there exists a simple closed curve J of*

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[*] *Ueber im kleinen zusammenhängende Kontinua,* Mathematische Annalen, vol. 98 (1927), p. 297.

[†] A set B is said to be an ϵ-*separation* of the point P in the set M if $M - B = A + C$, where A and C are mutually separated sets and A contains the point P, and further the diameter of the set $A + B$ is less than ϵ. See P. Urysohn, *Mémoire sur les multiplicités Cantoriennes,* Fundamenta Mathematicae, vol. 7 (1925), p. 65.

[‡] *On continuous curves in n dimensions,* this Bulletin, vol. 34 (1928), pp. 349–360, Theorem 1. For two quite similar theorems published at approximately the same time, see P. Alexandroff and L. Tumarkin, *Beweis des Satzes, dass jede abgeschlossene Menge positiver Dimension in einem lokal zusammenhängenden Kontinuum von derselben Dimension topologisch enthalten ist,* Fundamenta Mathematicae, vol. 11 (1928), pp. 141–144; and W. Stepanoff and L. Tumarkin, *Ueber eine Erweiterung abgeschlossener Mengen zu Jordanschen Kontinuen derselben Dimension,* Fundamenta Mathematicae, vol. 12 (1928), pp. 43–46. However for the present application we require the property (2) of the theorem by Whyburn and myself. This property is stated in neither of the latter two theorems.

[§] We consider a point as a special case of an arc.

diameter less than ϵ, enclosing P and such that $J \cdot M$ consists of a finite number of components, that is, every point of M may be ϵ-separated by a simple closed curve or a finite set of mutually exclusive arcs. If there exists a positive number r and a positive integer n such that, for every circle C with center P and radius less than or equal to r, the point P belongs to the boundary of exactly n $(n > 0)$ complementary domains of $M > C$, then P may be ϵ-separated in M by a set of n arcs, for any positive number ϵ.

If P belongs to the boundary of no complementary domain of M, then M contains a simple closed curve which encloses P and is of diameter less than ϵ^*. If P belongs to the boundary of a complementary domain D of M, let C be a circle with center P and radius less than $\epsilon/2$ and such that C contains a point of D. Now using a slight modification of the proof of Theorem 1 of our paper, *On the separation of points of a continuous curve by arcs and simple closed curves,*† we may complete our proof.

LEMMA B. *If the point P of the continuous curve M may be ϵ-separated, for any positive number ϵ, by a set S of n mutually exclusive continua such that P is not contained in a bounded complementary domain of any one, but not by any set S of less than n, then P may be ϵ-separated by a set of n mutually exclusive arcs.*

For each positive integer k, let C_k be a circle with center P and radius $1/k$ and let π_k be the number of complementary domains of $M + C_k$ that have P and a point of C_k on their boundaries. Evidently $\pi_{k+1} \geqq \pi_k$ for every k. If the number π_k increases indefinitely with k, then it is easy to see that the number of components of any ϵ-separating set must increase indefinitely as ϵ approaches zero. But by hypothesis, n components suffice. Hence there is an integer s such that for $t \geqq s$, $\pi_t = \pi_s$. First let us suppose that $\pi_s > 0$.

* G. T. Whyburn, *Concerning continua in the plane*, Transactions of this Society, vol. 29 (1927), p. 381, Theorem 7.

† Proceedings of the National Academy of Sciences, vol. 14 (1928), pp. 201–206.

It follows from Lemma A that P may be ϵ-separated by a set of π_s mutually exclusive arcs for any $\epsilon > 0$. By our hypothesis, $\pi_s \geq n$. And it is easy to see that for $\epsilon < 1/s$, no ϵ-separating set has less than π_s components. But P may be ϵ-separated by a set of n continua for each ϵ. Therefore $\pi_s \leq n$. Hence $\pi_s = n$ and our lemma is proved for the case $\pi_s > 0$.

Now consider $\pi_s = 0$. There are two cases according as P is or is not a limit point of $E_2 - M$. If P is not a limit point of $E_2 - M$, there exists a positive number r such that every point of the interior of a circle with center P and radius r belongs to M. Then for $\epsilon < r/2$ there is no ϵ-separating set S such that P is not contained in a bounded complementary domain of some one component of S. Hence this case is impossible. In the remaining case we shall show that n must be equal to one and complete our proof by establishing the following lemma.

LEMMA C. *If B denotes the point set consisting of the sum of the boundaries of the complementary domains of a continuous curve M, and P is a point of $\overline{B} - B$, then P may be ϵ-separated in M by a single arc, for any positive number ϵ.*

There exists a simple closed curve J_1 which belongs to M, is of diameter less than ϵ and contains P in its interior. As P belongs to $\overline{B} - B$ there exists a complementary domain D of M in the interior of J_1 and a positive number ϵ' such that no point of $J_1 + D$ is within a distance ϵ' of P. There exists a simple closed curve J_2 which belongs to M, is of diameter less than ϵ', and contains P in its interior.* Then J_2 lies in the interior of J_1 and D lies between J_1 and J_2. Now let J_3 be any simple closed curve which lies between J_1 and J_2 and contains a point of D. Let $G_1, G_2, G_3, \cdots$ be the set of all components of $J_3 \cdot (E_2 - M)$. This set contains at least one component and is countable. Each component G_i lies in a complementary domain D_i of M. Let B_i be the boundary of D_i and let

$$G = (J_3 - \sum G_i) + \sum B_i.$$

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* G. T. Whyburn, *Concerning continua in the plane*, loc. cit.

Now (1) G lies entirely between J_1 and J_2 except for points on J_1 and J_2, (2) M contains G, (3) G is a continuous curve, (4) P lies in a bounded complementary domain of G.* The outer boundary of this domain is a simple closed curve J_4†which belongs to M, contains P in its interior, and has two points X and Y on the boundary of some complementary domain D_i. Let XUY be an arc from X to Y which lies in D_i except for X and Y. The point P lies in the interior of J_4 and thus it lies in the interior of one of the two simple closed curves formed by the arc XUY plus one of the arcs of J_4 from X to Y. Call this arc XVY. Then XVY is the desired ϵ-separating arc, for P lies in the interior of the simple closed curve $XUY+XVY$ and this simple closed curve lies in the set consisting of J_1 plus its interior.

3. THEOREM 1. *In order that every arc of a continuous curve M be extendible in M in both directions it is necessary and sufficient that M contain no point which can be ϵ-separated in M by an acyclic continuous curve for every positive number ϵ.*

The condition is sufficient. Every "point" arc may be extended because every point of a continuous curve, except end points, is interior to an arc of M. But an end point may be ϵ-separated by a point for every ϵ, and there exists no such point by our condition. Now let α be any true arc of M and let P and Q be the end points of α. We shall show that α is extendible in M in the direction QP. Suppose this is not the case. Let us consider the components of $M-\alpha$. No one of these has P as a limit point for then α would be extendible in M in the direction QP, contrary to our supposition. Let ϵ be any positive number. There exists a point X of α so that the diameter of the subarc XP of α is less than ϵ, and there exists a point Y $(X \neq Y)$ of the subarc XP so that

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* For detailed proof of these and the following statements, see page 204 of my paper, *On the separation of points of a continuous curve by arcs and simple closed curves,* loc. cit.

† R. L. Moore, *Concerning continuous curves in the plane,* Mathematische Zeitschrift, vol. 15 (1922), pp. 254–260, Theorem 4.

the diameter of the subarc YP of α is less than $\epsilon/2$. Since only a finite number of the components of $M-\alpha$ are of diameter greater than any given positive number,* there exists a point $Z(Z \neq Y)$ on the subarc YP of α so that (1) no component of $M-\alpha$ of diameter greater than $\epsilon/2$ has a limit point on ZP and (2) no component of $M-\alpha$ has a limit point on both ZP and XQ. Then the subarc XZ is an ϵ-separating set of P in M. But this is contrary to the hypothesis. Hence α is extendible in M in the direction QP, and similarly in the direction PQ.

The condition is necessary. Suppose M contains a point P which can be ϵ-separated by an acyclic continuous curve for every ϵ. We shall show that this leads to a contradiction by constructing in M an arc which cannot be extended in M in one direction. By Lemma B, P may be ϵ-separated in M by an arc. There exists a sequence $J_1, J_2, J_3, \cdots$ of simple closed curves such that (1) the interior of J_i contains J_{i+1} and P, (2) the diameter of J_i approaches zero as i increases indefinitely, (3) $J_i \cdot M = A_i B_i$ consists of a single arc. When $A_i B_i$ is not a point, then A_i and B_i are the end points of $A_i B_i$. In the order from A_i to B_i let C_i and D_i denote the first and last limit points of the component of $M - A_i B_i$ that contains P. Let E_i and F_i $(i > 1)$ denote the first and last limit points on $A_i B_i$ of that component of $M - A_i B_i$ that contains $A_{i-1} B_{i-1}$. Each of the subarcs $C_i D_i$ and $E_i F_i$ of $A_i B_i$ is a cutting of M. The component of $M - C_i D_i$ containing P lies in the interior of J_i, and the component of $M - E_i F_i$ containing P lies in the interior of J_{i-1}. There exists an arc α with end points D_1 and D_2 and lying except for D_1 entirely in the component of $M - A_1 B_1$ that contains P.† In the order from D_1 to D_2 there is a first point G_1 of $A_2 B_2$ on α. Consider the two arcs $C_3 D_3$ and $E_3 F_3$. Either one end point of $E_3 F_3$ is not an interior point of $C_3 D_3$ or both end

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* W. L. Ayres, *Note on a theorem concerning continuous curves*, Annals of Mathematics, vol. 28 (1927), pp. 501–2.

† R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), p. 342.

points of E_3F_3 are interior points of C_3D_3. In the first case let H_1 be this end point. And in this case there is an end point H_2 of C_3D_3 so that the subarc H_1H_2 of A_3B_3 contains the entire arc C_3D_3. In the latter case let H_1 be either end point of E_3F_3 and let H_2 be that end point of C_3D_3 so that the subarc H_1H_2 of A_3B_3 contains the entire arc E_3F_3. In either case, then, we have a subarc H_1H_2 of A_3B_3 so that H_1H_2 contains either C_3D_3 or E_3F_3. There exists* an arc G_2H_1 of M which lies, except for one end point G_2 on J_2 and one end point H_1 on J_3, entirely between J_2 and J_3. In general, the arc $A_{2n+1}B_{2n+1}$ ($n = 1, 2, 3, \cdots$) contains an arc $H_{2n-1}H_{2n}$ which contains one of the two arcs $C_{2n+1}D_{2n+1}$ or $E_{2n+1}F_{2n+1}$, and is such that H_{2n-1} is either E_{2n+1} or F_{2n+1} and H_{2n} is either C_{2n+1} or D_{2n+1}. There exist two arcs $G_{2n}H_{2n-1}$ and $H_{2n}G_{2n+1}$ of M such that G_{2n} is a point of J_{2n}, G_{2n+1} is a point of J_{2n+2}, and these two arcs lie except for their end points entirely between J_{2n} and J_{2n+1} and J_{2n+1} and J_{2n+2} respectively. Let

$$\beta = P + C_1D_1 + \text{subarc } D_1G_1 \text{ of } \alpha + \sum_{n=1}^{\infty}(\text{subarc } G_{2n-1}G_{2n} \text{ of } A_{2n}B_{2n})$$
$$+ \sum_{i=1}^{\infty}H_{i+1}G_{i+1} + \sum_{n=1}^{\infty}(\text{subarc } H_{2n-1}H_{2n} \text{ of } A_{2n+1}B_{2n+1}).$$

Evidently β is an arc of M from C_1 to P. And further we may show that this arc may not be extended in M in the direction C_1P for each of the arcs $H_{2n-1}H_{2n}$ contains a cutting of M and these cuttings of M are such that the diameter of the component of $M - H_{2n-1}H_{2n}$ containing P approaches zero as n increases. Hence any arc of M with P as an end point contains a point of infinitely many of the arcs $H_{2n-1}H_{2n}$. Therefore β cannot be extended in M in the direction C_1P.

4. THEOREM 2. *In order that every arc of the continuous curve M be extendible in M in both directions it is necessary and sufficient that* (1) *the sum B of the boundaries of all the complementary domains of M be a closed point set*, (2) *every point of B be either a cut point of M or belong to the boundaries of at least two complementary domains of M.*

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* See R. L. Wilder's paper cited in the footnote on p. 855.

The conditions are necessary. Suppose $\overline{B}-B$ contains a point P. By Lemma C, P may be ϵ-separated in M by an arc for every positive ϵ. But by Theorem 1, M contains no such point P. Therefore $\overline{B}-B$ is vacuous and the first condition is necessary. Now suppose P is any point of B. If C is any circle with center P, then P is accessible from at least one complementary domain of $M+C$ since B contains P. If for every circle C, P is accessible from just one complementary domain of $M+C$, then P may be ϵ-separated, for any $\epsilon>0$, by an arc, from Lemma A. But this is impossible by Theorem 1. Hence there is some circle C so that P is accessible from at least two complementary domains D_1 and D_2 of $M+C$. If D_1 and D_2 belong to different complementary domains of M, we have the latter part of condition (2). And if D_1 and D_2 belong to the same complementary domain of M, it may be shown that P is a cut point of M, which is the first part of condition (2).

The conditions are sufficient. Let P be a point of M. If P is not a point of B, it is not a point of $\overline{B}$ by condition (1) and there exists a circle with center P and radius r such that every point of the interior of the circle belongs to M. Then P cannot be ϵ-separated by an acyclic continuous curve for $\epsilon<r$. If P belongs to the boundary of just one complementary domain of M, P is a cut point of M and for ϵ sufficiently small, no cut point may be ϵ-separated in M by a set consisting of just one component. And if P belongs to the boundaries of two complementary domains of M, we may show that any ϵ-separating set, for ϵ sufficiently small, must contain more than one component. Hence M contains no point which can be ϵ-separated in M, for every positive ϵ, by an acyclic continuous curve. By Theorem 1, every arc of M is extendible in M in both directions.

5. THEOREM 3. *If every arc of the continuous curve M is extendible in M in both directions, every point of M lies on the boundary of some complementary domain of M ($M \equiv B$).*

By Theorem 2, the set B, the sum of the boundaries of the

complementary domains of M, is a closed point set. Now suppose there is a point P of $M - B$. Every complementary domain of M is a complementary domain of the point set B and in addition there is at least one complementary domain of B, all of whose points belong to M. Let D denote the complementary domain of B containing P. We shall assume that D is bounded for we may always produce this by an inversion. Let B_1 denote the outer boundary of D.* Then B_1 is the common boundary of two domains, one the exterior of B_1 and the other a bounded domain containing D.† Then no point of B_1 disconnects B_1.‡ Then by a theorem due to R. L. Moore,§ only a countable number of the points of B_1 are cut points of M. Hence, by Theorem 2, uncountably many points of B_1 are accessible from two complementary domains of M. Each of these points is also a limit point of D. There are only a countable number of complementary domains of M and thus only a countable number of pairs of complementary domains of M. And by a theorem due to G. T. Whyburn,¶ at most two points can be accessible from each of the two domains of a pair and at the same time be limit points of D. Thus in all only a countable number of points of B_1 can be accessible from two complementary domains of M. Therefore our assumption that $M - B$ is non-vacuous is false.

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* R. L. Moore, loc. cit., p. 256.

† Ibid., Theorem 3, p. 258.

‡ R. L. Moore, *Concerning the common boundary of two domains*, Fundamenta Mathematicae, vol. 6 (1924), pp. 203–213, Theorem 2.

§ *Concerning the cut-points of continuous curves and of other closed and connected point-sets*, Proceedings of the National Academy of Sciences, vol. 9 (1923), pp. 101–106, Theorem B*.

¶ *Concerning plane closed point sets which are accessible from certain subsets of their complements*, Proceedings of the National Academy of Sciences, vol. 14 (1928), pp. 657–666, Theorem 1 and lines 1–3 of page 659.