PICARD ON PARTIAL DIFFERENTIAL EQUATIONS

Leçons sur quelques Types Simples d'Equations aux Dérivées Partielles avec des Applications à la Physique Mathématique. By Émile Picard. Paris, Gauthier-Villars, 1927. i+214 pp.

This book forms the first volume of a new mathematical series, *Cahiers Scientifiques,* edited by G. Julia. From the preface to volume 3 of the series, one learns with regret that Picard has abandoned the project of completing volume 4 of his *Traité d'Analyse,* which was to give the theory of partial differential equations. Instead, the subject matter of this volume will be presented in several volumes of *Cahiers Scientifiques.* The book under review contains lectures given at the Sorbonne in 1907 and repeated, with some additions, in 1925.

The first four lectures deal with the simplest and best known of the equations of parabolic type, namely Fourier's equation

$$\partial u/\partial x = \partial^2 u/\partial y^2.$$

Lecture 1 begins with the existence theorem, shows that non-analytic solutions are possible for analytic initial data, and derives Fredholm's example of a power series whose circle of convergence is a natural boundary. Lecture 2 derives the fundamental solution, shows how this is used to solve the boundary problem $u=f(x)$ for $t=0$, and as an application gives Weierstrass' original proof of his theorem on the approximation to a continuous function by polynomials. Lecture 3 applies the preceding results to various boundary problems in heat conduction, and Lecture 4 gives an account of Lord Kelvin's investigation of the propagation of an electric current along a cable.

Lectures 5 to 11 deal with various aspects of the theory of integral equations. In Lecture 5, a brief account of the Fourier integral is given and applied to an integral equation which was set up in Lecture 3. Lecture 6 deals with the properties of the integrals

$$\int_0^\infty \cos xy\phi(y)dy \text{ and } \int_0^\infty e^{-xy}\phi(y)dy$$

considered as functions of x. In Lecture 7, an integral equation of the first kind is set up for the solution of Dirichlet's problem (in two dimensions) by a potential of a single layer on the boundary curve, and the main results of Fredholm on integral equations of the second kind are enumerated.

In Lecture 8, the singular integral equations of the second kind

$$f(x) + \lambda \int_0^\infty \cos x\xi f(\xi)d\xi = \phi(x)$$

and

$$f(x) + \lambda \int_{-\infty}^\infty e^{-|x-\xi|} f(\xi)d\xi = \phi(x)$$

are briefly considered. Lecture 9 deals with the integral

$$\frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-x},$$

where $f(z)$ is defined only on the closed curve C (so that $f(z)$ is not necessarily analytic), and the principal value of the integral, when x approaches C, is defined. Lecture 10 raises the question of the necessary and sufficient condition to be imposed on $f(z)$ on C in order that the above integral shall be an analytic function (which is then $=f(x)$ by Cauchy's formula). The condition is

$$f(x) = \frac{1}{\pi i} \text{ Princ. val. } \int_C \frac{f(z)}{z-x}dz$$

for every x on C. Lecture 11 applies the principal values to the solution of Riemann's problem of finding two analytic functions one regular inside C and the other outside, such that their ratio shall take prescribed values on C. Hilbert's problem of determining an analytic $f(z)=u(x, y)+i\,v(x, y)$ such that $au+bv+c=0$ on C, where a, b, and c are functions of position on C, is also considered, and the lecture closes with a discussion of the tangential derivative of a single layer potential. With Lecture 12, the author returns to partial differential equations and gives a brief survey of Cauchy's problem and the theory of characteristics. Lectures 13 to 16 deal mainly with linear equations of the hyperbolic type, and the method of successive approximations is used to solve the following boundary problems: (1) the values of u are given on two characteristics; (2) u and one of its first derivatives are given along a curve which is not a characteristic (Cauchy's problem); (3) u is given on a characteristic and also on a non-characteristic; (4) u given on two non-characteristics. In Lectures 17 and 18, the problems of Lectures 13 to 16 are treated by Riemann's integration method. This method is applied to the equation of a vibrating string and to the telegraph equation in Lecture 19. Lecture 20 introduces a method due to Poincaré for the telegraph equation, and this gives occasion for some remarks on integral equations of the first kind, particularly that of Abel.

The author now passes to linear equations of the second order and elliptic type. Lecture 21 gives the proof of the analyticity of the solutions of the general second-order equation of the elliptic type, and Lecture 22 presents some results on the rather delicate question of the existence of the normal derivative of the solution. Lecture 23, which is rather loosely connected with the preceding ones, finally gives some properties of the spherical potential of Boussinesq.

Many results and still more proofs in this book are new, and the exposition has all the qualities of lucidity and elegance which one is accustomed to expect from its distinguished author.

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