DE VRIES ON FOUR DIMENSIONS

Die vierte Dimension, eine Einführung in das Vergleichende Studium der Verschiedenen Geometrien. By Dr. H. K. de Vries. Translated into German from the second Dutch edition by Frau Dr. Ruth Struik. Wissenschaft und Hypothese, XXIX, Leipzig and Berlin, Teubner, 1926. ix+167 pp.

Although there is no mention of non-euclidean geometry in the title, nor even on the title-page, nearly half of the book is devoted to it. In fact, we have here two books put together under the title of the first. The books are independent: either could have been put first or published alone. Only in one place in the second (pages 139–141) is there introduced a little of the geometry of four dimensions, and at the very end there is an outline of the rise of this geometry, a little more than two pages, that might have been put at the end of the first part.

The treatment of the book is not strictly mathematical. We have rather a popular exposition, although considerable mathematical detail is suggested in the explanations, giving some idea of the way in which these subjects have been developed as well as the mere results of their development.

A small book, not too condensed, must leave out many things. Thus we find nothing about hyperprisms, hyperpyramids, hypercylinders, and hypercones, and nothing about hypervolume. Some might have chosen these and given only a paragraph to the regular polytopes or hypersolids. Many details of the non-euclidean geometry are presented very briefly. But the choice of material is in general excellent, and the book is well planned to impart some knowledge of these two branches of mathematics.

In both parts of the book much is said about the historical development of the subjects considered, about their usefulness, and about the frame of mind in which the reader should study them, and many interesting remarks are thrown in. Thus of the value of extending the chain of dimensions beyond the third we are told (page 22) that the notion of value is in high degree relative, and what is of inestimable value for one man leaves another entirely cold. But for the mathematician and for everyone who has an interest in scientific thought, it is of great value to know that a four or five or n dimensional geometry is possible, and the mathematician will not be satisfied to know that it is possible, but will want to know more exactly about it, as always when a new field of investigation appears.

At the beginning the relations of the simpler elements, lines, planes, and hyperplanes, their determinations and intersections, parallelism and perpendicularity, and the various kinds of angles formed by them, are gone into very thoroughly and carefully, and this study is the best sort of preparation even for a superficial understanding of the more complicated figures and relations of hyperspace.

The notion of point-value of a space is most helpful, for the dimension-number d is 1 less than w, the number of points necessary to determine a given space, and some formulas are obtained more easily by considering w. Thus two spaces, R_{d_1} and R_{d_2}, having in common a space $R_{d_{12}}$, determine a space R_d, the point-values of these spaces being connected by the relation $w = w_1 + w_2 - w_{12}$ and therefore their dimension-numbers by the relation $d = d_1 + d_2 - d_{12}$, which may also be used to determine the dimension of the intersection of two given spaces lying in and determining another given space. After this formula has been applied to spaces in R_3 and in R_4, it is suggested that the reader experiment with R_5 and R_6, and then we are told (page 35) that he will not be able to suppress the feeling of having extended very widely his horizon, and that this is the reward for the striving necessary at the beginning.

Turning to the second part, the non-euclidean geometry, we find the quadrilateral of Saccheri and his three hypotheses taken as the starting-point, with a brief indication of the results to which these three hypotheses lead. There is also an account of the way in which the hyperbolic and elliptic geometries are represented as the geometries on the pseudo-sphere and sphere, and by projection on a plane. The *boundary-curve* or *horocycle*, and the *equidistant-curve* are described (page 150) as new figures of hyperbolic geometry, and the polar relation of lines and points and of planes and points in the elliptic geometry (page 130), while the two kinds of elliptic geometry, the double and simple elliptic geometries, are mentioned (page 133). But the treatment of all of these details is very brief.

We close with an outline of the contents of this book.

After an introduction of about three pages we have Part First, euclidean geometry of several dimensions [chiefly of four dimensions], pages 4–92, and Part Second, non-euclidean geometry, pages 92–165.

The first part may be divided roughly as follows:

Points, lines, planes, and spaces, their intersections, point-values, etc., pages 4–35.

Infinitely distant elements and parallelism, pages 35–45.

Perpendicularity and rotation, the hypersphere, pages 45–59.

Angles, the number of angles between two spaces, the two angles between two planes, pages 60–77.

The regular polytopes, pages 77–92.

An outline of the second part, with some changes in order, will be:

The postulates of Euclid and the three hypotheses of Saccheri, pages 92–112.

The sum of the angles of a triangle and its area, pages 112–116, 153–161.

Euclidean geometry, pages 116–120.

Hyperbolic geometry, pages 120–125, 145–153.

Elliptic geometry, pages 125–134.

Geometrical representations, pages 134–145, 161–163.

And finally, a historical paragraph in regard to geometry of n dimensions, pages 163–165.

H. P. MANNING