SINGULAR MANIFOLDS AMONG THOSE OF AN ANALYTIC FAMILY*

BY O. D. KELLOGG

1. *Exceptional Occurrence of Singular Manifolds.* This note is concerned with the following theorem.

Let R denote a closed region, consisting of an open continuum of the space of the n complex variables $z_1, z_2, \cdots, z_n$, together with its boundary points. Let the functions $F_i(z_1, z_2, \cdots, z_n)$, $i=1, 2, \cdots, m$, $m \leqq n$, be analytic at all points of R. Let M_k denote the matrix

$$(M_k) \qquad \left\| \frac{\partial F_i}{\partial z_j} \right\|, \qquad \begin{matrix} (i = 1,2, \cdots, k), \\ (j = 1,2, \cdots, n). \end{matrix}$$

We assume that M_m is of rank m at some point of R.

Consider the manifold defined by the equations

$$(A) \quad F_1(z_1,z_2, \cdots, z_n) = c_1, \cdots, F_m(z_1,z_2, \cdots, z_n) = c_m,$$

where $c_1, c_2, \cdots, c_m$ are complex constants.

For all but a finite number of values of c_1 the manifold defined by the first equation (A) contains no points in R at which the rank of M_1 is less than 1.

If $c_1, c_2, \cdots, c_k$ have been chosen so that the matrix M_k is of rank k at every point in R on the manifold defined by the first k equations (A), then for all but a finite number of values of c_{k+1} the manifold defined by the first $k+1$ equations (A) contains no points in R at which the rank of the matrix M_{k+1} is less then $k+1$.

Thus, if $c_1, c_2, \cdots, c_m$ are chosen in order, each avoiding a certain finite set of values, the manifold defined by the equations (A) will have no singular points in R.†

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† As far as I know, a proof of this general theorem has not been published. Birkhoff and I gave it for the case in which the functions F_i are polynomials (Transactions of this Society, vol. 23 (1922), pp. 97–98), and

By a singular point of a manifold, we mean a point at which a unique tangent linear manifold of the same order does not exist. A sufficient condition that a point of the manifold defined by the equations (A) be not singular is that the rank of the matrix M_m is m at that point. That the condition is not necessary may be seen by the example

$$z_1^2 z_2^2 - z_3^3 = 0, \qquad z_1^2 z_2^2 + z_3^3 = 0.$$

The manifold defined by these equations consists of the two intersecting straight lines $z_1 = z_3 = 0$ and $z_2 = z_3 = 0$. Its only singular point is at the origin, yet the rank of M_2 is less than 2 at every point of the manifold.

It is not stated in the theorem that the values of the c_j characterizing singular manifolds are finite in number. In general, the value of c_{k+1} to be avoided, if the resulting manifold is to be non-singular, is a function of the c_j previously chosen. Thus in the case

$$z_1 - z_2 z_3 = c_1, \qquad z_1 = c_2,$$

the value of c_2 to be avoided is c_1. What is the case, is that among the ∞^{2m} members of the family defined by (A), there are at most ∞^{2m-2} singular manifolds, a precise interpretation of this statement being given in the theorem.

2. *Needed Results in the Theory of Analytic Functions.* We attach our considerations to the study of simultaneous equations in the second volume of Osgood's *Lehrbuch der Funktionentheorie*,* where we find a theorem, of which the part we need may be stated as follows:

Given a system of simultaneous equations

$$G_1(z_1, z_2, \cdots, z_n) = 0, \cdots, G_l(z_1, z_2, \cdots, z_n) = 0,$$

where $G_k(z_1, z_2, \cdots, z_n)$, $k = 1, 2, \cdots, l$, *is analytic at the point* $(a) = (a_1, a_2, \cdots, a_n)$ *and vanishes there, but does not*

Morse gives a proof for the case $m = 1$ in a paper in the April number of the American Journal of Mathematics, vol. 51 (1929). The theorem is apparently useful; in addition to its role in the two papers just cited it has served me at several points in a forthcoming book on Potential Theory.

* Leipzig, 1st ed., 1924, 2d ed., 1929, Chap. II, §17.

vanish identically. The solutions of this system in the neighborhood of (a) *consist either of the point* (a) *alone, or of one or more manifolds* g. The context makes it clear that the number of these manifolds is *finite.*

The essential property of the manifolds g for our purposes is that *any two points of any one of them can be connected by a curve* γ, *lying in the manifold, and analytic except at a finite number of points.*

We begin, in establishing this property, with a pseudo-algebraic manifold $\mathfrak{G}$ defined by the irreducible equation

$$(1)\quad H(w, u_1, u_2, \cdots, u_\rho) = w^\mu + E_1 w^{\mu-1} + \cdots + E_\mu = 0$$

and the inequalities $|u_i| < h$, where h is a suitably restricted positive number. The coefficients E_j are power series in the variables u_i convergent for $|u_i| < h'$, $h' > h$, vanishing at the origin $O(u_1 = u_2 = \cdots = u_\rho = 0)$.

If $P'(w', u_1', u_2', \cdots, u_\rho')$ is any point of $\mathfrak{G}$, and if in the equation (1) we set $u_1 = u_1' t$, $u_2 = u_2' t$, $\cdots$, $u_\rho = u_\rho' t$, the resulting equation defines w as an algebroid function of t. If the latter equation is reducible, we select the irreducible factor of the left hand member which vanishes for $w = w'$ when $t = 1$, and equate it to 0. The discriminant of the resulting equation vanishes at most a finite number of times on the real interval (0, 1), and hence from the branches of the function w of t there can be selected a (complex) single-valued function $w = f(t)$, which is continuous on the closed interval (0, 1), analytic except at a finite number of points, and which reduces to 0 for $t = 0$, and to w' for $t = 1$. Then the curve

$$(\gamma')\qquad w = f(t), \quad u_1 = u_1' t, \cdots, \quad u_\rho = u_\rho' t, \quad 0 \leqq t \leqq 1,$$

lies in $\mathfrak{G}$, connects P' with O, and is analytic except at a finite number of points. By means of a curve consisting of two such parts, any two points P' and P'' of $\mathfrak{G}$ can be connected.

We now consider a set of functions $w_1, w_2, \cdots, w_\sigma$ of

$u_1, u_2, \cdots, u_\rho$, belonging to the pseudo-algebraic manifold $\mathfrak{G}$, that is, having the following properties: they are one-valued and continuous on $\mathfrak{G}$, and analytic at every point of $\mathfrak{G}$ at which the discriminant of the equation (1) does not vanish; the function $\alpha_1 w_1 + \cdots + \alpha_\sigma w_\sigma$ of the u_i has, for non-specialized values of the parameters $\alpha_1, \cdots, \alpha_\sigma$, branches, corresponding to the branches of w, no two of which are identically equal. Then the set of points of the space of $(w_1, w_2, \cdots, w_\sigma, u_1, u_2, \cdots, u_\rho)$ which correspond to the points $(w, u_1, \cdots, u_\rho)$ of $\mathfrak{G}$ constitutes a manifold g. The most general manifold g is either one defined in this way, or else one derived from it by a non-singular linear transformation of the variables. As such a transformation does not change the essential character of the curve γ, we may confine ourselves to the manifolds g as first defined.

The functions $w_1, w_2, \cdots, w_\sigma$ are pseudo-algebraic, and as they are one-valued and continuous on $\mathfrak{G}$, they assume at O unique values $w_{10}, w_{20}, \cdots, w_{\sigma 0}$. If $p'(w_1', w_2', \cdots, w_\sigma', u_1', u_2', \cdots, u_\rho')$ is any point of g, there corresponds to it a unique point P' of $\mathfrak{G}$, which is connected with O by a curve γ'. Just as the function $w=f(t)$ was determined, we may determine from among the branches of w_1, a single-valued function $w_1=f_1(t)$, continuous on the interval $(0, 1)$, analytic, except at a finite number of points, and reducing to w_{10} for $t=0$ and to w_1' for $t=1$. When this and similar functions have been determined for $w_1, \cdots, w_\sigma$, we shall have a curve

$$(\gamma)\ w_1 = f_1(t), \cdots, w_\sigma = f_\sigma(t), u_1 = u_1' t, \cdots, u_\rho = u_\rho' t, \quad 0 \leqq t \leqq 1,$$

from two of which can be constructed a curve of the required character, connecting any two given points of g.

3. *Proof of the Theorem.* Let (a) be any point of R. Then in the neighborhood of (a), the solutions of the equations

$$(2) \qquad \frac{\partial F_1}{\partial z_1} = 0, \quad \frac{\partial F_1}{\partial z_2} = 0, \cdots, \frac{\partial F_1}{\partial z_n} = 0,$$

(not all of which are identically satisfied, since M_m is of rank m at some point of R), if there are any, consist, by Osgood's theorem, either of the point (a) alone, or of a finite number of manifolds g. Along an analytic arc of a curve γ joining any two points of such a manifold, $dF_1=0$. It follows that F_1 is constant on each such manifold, so that the number of values of F_1 corresponding to solutions of the equations (2) in the neighborhood of (a), is finite. By the Heine-Borel theorem, the same is true for the whole region R. Hence for all but a finite number of values of c_1 the manifold defined by the first equation (A) contains no points in R at which M_1 is of rank less than 1. The first part of the theorem is thus established.

Suppose now that $c_1, c_2, \cdots, c_k$ have been chosen so that at every point in R of the manifold defined by the first k equations (A), the matrix M_k is of rank k. We consider the simultaneous system of equations consisting of the two sets

$$(3)\ F_1(z_1,z_2,\cdots,z_n)-c_1=0,\ \cdots, F_k(z_1,z_2,\cdots,z_n)-c_k=0,$$

and

$$(4)\ D_1(z_1,z_2,\cdots,z_n)=0,\ \cdots,\ D_r(z_1,z_2,\cdots,z_n)=0,$$

where $D_1, \cdots, D_r$ are all the determinants of order $k+1$ formed from the matrix M_{k+1}, except such as may be identically 0. Not all vanish identically, since, by hypothesis, M_m is of rank m at some point of R.

If this system has any solutions in the neighborhood of a point (a) of R, they consist, by Osgood's theorem, of the point (a) alone, or of a finite number of manifolds g. Let g denote such a manifold. On it M_k and M_{k+1} are everywhere of rank k. Hence from the equations $dF_1=dF_2=\cdots=dF_k=0$, which follow from (3), we infer that $dF_{k+1}=0$ at all points of the curve γ connecting any two given points of g. By reasoning previously employed, it follows that for all but a finite number of values of c_{k+1}, the matrix M_{k+1} is of rank $k+1$ at all points of R on the manifold defined by the first $k+1$ equations (A). This proves the second part of the theorem.

The truth of the final statement of the theorem emerges when to k are assigned successively the values $1, 2, \cdots, m-1$, and when it is recalled that a sufficient condition that the manifold defined by the equations (A) has no singular points in R is that the matrix M_m is of rank m at all points of the manifold in R.

HARVARD UNIVERSITY

ON THE FOUNDATIONS OF GENERAL INFINITESIMAL GEOMETRY*

BY HERMANN WEYL

In connection with a seminar on infinitesimal geometry in Princeton, in which I took part, it seemed desirable to clarify the relations between the work of the Princeton school and that of Cartan.

With a group $\mathfrak{G}$ of transformations in m variables ξ is associated, in accordance with Klein's Erlanger Program, a homogeneous or plane space $\mathfrak{R}$ of the kind $\mathfrak{G}$; a point of $\mathfrak{R}$ is represented by a set of values of the "coordinates" ξ^α and figures which go into each other on subjecting the coordinates to a transformation of $\mathfrak{G}$ are to be considered as fully equivalent. The transformations of $\mathfrak{G}$ give at the same time the transition between two allowable "normal" coordinate systems in $\mathfrak{R}$. If we have two spaces $\mathfrak{R}$, $\mathfrak{R}'$ of the kind $\mathfrak{G}$ and set up a definite normal coordinate system in each of them, then such a transformation can be interpreted as an isomorphic representation of $\mathfrak{R}$ on $\mathfrak{R}'$. $\mathfrak{G}$ is assumed to be transitive.

Cartan† developed a general scheme of infinitesimal geometry in which Klein's notions were applied to the tangent plane and not to the n-dimensional manifold M itself. The

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† E. Cartan, *Sur les variétés à connexion affine et la théorie de la relativité généralisée*, Annales de l'École Normal Supérieure, vol. 40 (1923), pp. 325–412; in particular, p. 383, etc.