NON-ISOLATED CRITICAL POINTS OF FUNCTIONS*

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The isolated critical points of real functions of n independent real variables have been treated in an elegant manner by Marston Morse.‡ This treatment obtains definite relations between the numbers of critical points of $n+1$ types that appear in a bounded portion of the space of the independent variables. Morse requires his functions to have continuous third partial derivatives in the neighborhoods of the critical points and imposes conditions that are sufficient to insure the existence of at most a finite number of such points in the domain under consideration. In the present note§ we consider functions that have continuous first partial derivatives and may have an infinite number of critical points, or even continua of such points, in the given domain. As a special case of our results we obtain the *minimax principle* of Birkhoff in the modified form given by Bieberbach.‖

Let S denote the space of the n real variables $x_1, x_2, \cdots, x_n$, and let $(x)=(x_1, x_2, \cdots, x_n)$ denote a point in this space. Let R be a bounded,¶ open, and connected point set in S and let C, the boundary of R, be connected. Let the real function $f(x_1, \cdots, x_n)$ be single-valued and continuous

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‡ Transactions of this Society, vol. 27 (1925), pp. 345–396.

§ The author is indebted to Marston Morse for helpful suggestions concerning the material of this paper.

‖ *Differentialgleichungen*, Berlin, Springer, 1927, p. 140.

¶ The terms of classical point set theory are used in their usual sense. The distance between two points is given by a generalization of the ordinary distance formula in the plane. If K denotes a point set, then K' is used to denote K together with all of its limit points.

throughout R' and furthermore let $f_i(x_1, \cdots, x_n) = \partial f/\partial x_i$, $(i=1, \cdots, n)$, be continuous in R'. On C, let $f(x) = K$, a constant, and at all points of $R = R' - C$ let $f(x)$ be less than K.

A point of R at which $f_1(x) = f_2(x) = \cdots = f_n(x) = 0$ is called a critical point of $f(x)$ and the value of $f(x)$ at such a point is called a critical value of $f(x)$. Let H denote the set of all critical points of $f(x)$ in R.

THEOREM 1. *For each point p of H there exists a connected subset $M(H, p)$ of H that contains p and has $f(x) = f(p)$ at each of its points and finally, such that $M(H, p)$ is not a proper subset of any other subset of H that is connected, contains p, and has $f(x) = f(p)$ at each of its points.*

PROOF. Let p be a point of H and let T denote the set of all points of H for which $f(x) = f(p)$. If T is connected, it is the desired set. Since T is bounded, we may find a hypercube C_1 of edge d containing T on its interior. Let $C_1, C_2, \cdots$ denote the set of hypercubes obtained by dividing C_1 into 2^n hypercubes $C_2, C_3, \cdots, C_{2^n+1}$ of edge $d/2$, then dividing each of these into hypercubes of side $d/4$ and continuing this subdivision indefinitely. We suppose these arranged so that for every i, C_i has its edge greater than or equal to the edge of C_{i+1}. Let C_{n_1} be the cube of lowest subscript that contains a point of T and is such that T can be expressed as the sum of two mutually exclusive closed sets P_1 and Q_1 such that P_1 contains p but no point of C'_{n_1}. Certainly such a cube exists since, by hypothesis, T is not connected and hence can be expressed as the sum of two mutually exclusive closed sets S_1 and S_2. There exists a positive number ϵ such that the sets S_1 and S_2 are at a distance apart greater than ϵ and hence any cube of the set $C_1, C_2, \cdots$, that has its edge of length less than $\epsilon/2$ and contains a point of the set which fails to contain p may be chosen. If P_1 is not connected, we let C_{n_2} be the hypercube of the set $C_1, C_2, \cdots$, of lowest subscript which contains points of P_1 and has the property that P_1 can be expressed as the sum of two mutually exclusive sets P_2 and Q_2 such that P_2 contains p but contains no point of C'_{n_2}.

This process may be continued indefinitely unless one of the P's, P_j, is connected. If P_j is connected, it is the desired set since $T-P_j=Q_1+Q_2+\cdots+Q_j$ and hence P_j cannot be a proper subset of a connected subset of T. If the process continues indefinitely, then the sequence $P_1, P_2, \cdots$, has at least one common point, namely, p. Let M denote the totality of such common points. M must be connected, since otherwise it would be the sum of two mutually exclusive closed sets (since M is closed). These sets would be at a distance apart greater than a definite positive constant and hence we could find a hypercube C_k of the set $C_1, \cdots$, containing points of that set which failed to contain p, but having no point of the other set in C_k'. This, however, is impossible since by our method of picking $C_{n_1}, \cdots, C_k$ would have been chosen at at least the kth stage. Since $T=M+\sum_{i=1}^{\infty} Q_i$, it follows that any point q of $T-M$ belongs to one of the Q's, Q_q, and since M belongs to P_q, it follows that q could not belong to a connected subset of T that contains M. Hence M is the desired set $M(H, p)$.

DEFINITION. *By the critical sets of* $f(x)$ *in* R *we mean the sets* $M(H, p)$ *of Theorem* 1.

THEOREM 2. *Let* $K_1, K_2, \cdots, K_m$, *where* m *is a positive integer, be subsets of* H *and let each* K_i *have the property that any two of its points can be joined in* K_i *by a rectifiable Jordan arc* of finite length. If* $k=K_1'+\cdots+K_m'$ *is connected,* $f(x_1, \cdots, x_n)$ *is constant on* k.

PROOF. A theorem on rectifiable curves in the plane and in three-space can readily be extended to hold in n-space.† This theorem states that if $x=f_1(t)$, $y=f_2(t)$, $z=f_3(t)$ is a rectifiable curve of finite length, then df_1/dt, df_2/dt, df_3/dt exist and are finite at every point of the t interval with the possible exception of a set of points of measure zero.

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* The definition of such an arc is an obvious extension of the two-dimensional definition.

† Hobson, *Theory of Functions of a Real Variable*, Cambridge, 1921, vol. I, pp. 320 and 376.

Let p_1 and p_2 be any two points of K_i and let $x_j=f_j(t)$, $j=1, \cdots, n$, where t denotes the length of arc from p_1, be a Jordan arc of the type whose existence is conditioned in the theorem. We have $df=\sum_{j=1}^{n} f_j dx_j$ *almost everywhere* on the t interval and since the arc is a subset of H, we have $df=0$ *almost everywhere* on this arc. An application of Lebesgue integration yields $f(p_1)=f(p_2)$. Hence $f(x)$ is constant on K_i. From continuity, it is therefore constant on K_i' and from the connectedness of k it follows that $f(x)$ has a fixed constant value on this set of points.

DEFINITION. *A critical set $M(H, p)$ is said to be a minimal set of $f(x)$ provided it is true that for every point q of $M(H, p)$ there exists a positive number ϵ_q such that $f(x) \geqq f(q)$ for every point (x) whose distance from q is less than ϵ_q, the equality sign holding only for points of $M(H, p)$.*

THEOREM 3. *If $f(x_1, \cdots, x_n)$ has two distinct minimal sets M_1 and M_2 in R, there exists at least one critical point of $f(x_1, \cdots, x_n)$ in R that does not belong to M_1+M_2.*

PROOF. Let the notation be chosen so that if m_1 and m are the values of $f(x)$ on M_1 and M_2, respectively, then $m \geqq m_1$. Let the notation $G(f \leqq h)$ mean the subset of R' on which $f \leqq h$, and let $G(f=h)$ mean the subset of R' on which $f=h$, while $G(f<h)$ means the subset of R' on which $f<h$. In $G(f \leqq m)$ there are two closed, connected, and mutually exclusive sets N_1 and N_2 such that N_1 contains M_1, $N_2 \equiv M_2$, and N_1 and N_2 are the maximal connected subsets* of $G(f \leqq m)$ that contain M_1 and M_2, respectively. That N_1 and N_2 are mutually exclusive follows when one notes that if they had a point q in common it would follow from the connectedness of N_1 and N_2 that this point would belong to M_2 and be a limit point of points of $N_1 - M_1$ (since M_1 is closed), that is, points

* By a maximal connected subset of a set T having certain properties we understand a subset which has the properties and is connected but which is not a proper subset of any other subset of T having these properties. The existence of such sets for each of our applications is established by a repetition of the arguments used in proving Theorem 1 with slight verbal modifications.

of $G(f<m)$. This is impossible since M_2 is a minimal set of $f(x)$.

Let h be any number such that $m \leqq h \leqq K$ and let N_{1h} and N_{2h} be the maximal connected subsets of $G(f \leqq h)$ that contain M_1 and M_2, respectively. For $h=m$, N_{1h} and N_{2h} have no point in common while for $h=K$, these two sets have a common point and are therefore identical. An application of the Dedekind-cut postulate yields a number k, $m \leqq k \leqq K$, such that k is either the greatest number for which N_{1k} and N_{2k} have no point in common or it is the least number for which $N_{1k} \equiv N_{2k}$. We show that k is the least number for which $N_{1k} \equiv N_{2k}$. Suppose k is the greatest number for which N_{1k} and N_{2k} are mutually exclusive. Noting that $R' = G(f \leqq K)$ is connected, we can and will choose a sequence $k_1 > k_2 > \cdots > k_i > \cdots$, so that $k < k_i < K$ and $\lim_{i \to \infty} k_i = k$. $N_i = N_{1k_i} = N_{2k_i}$, $i = 1, 2, \cdots$, is a sequence of closed, bounded, and connected sets having the property that for each i, N_{i+1} is a subset of N_i. A direct generalization of the usual proof in the plane, making use of the Heine-Borel property and the set of hypercubes introduced in the proof of Theorem 1, shows that the common part, $N_{1k} + N_{2k}$, of this sequence is closed and connected. This, however, contradicts the hypothesis that k was a number for which the sets N_{1k} and N_{2k} have no common point and shows that k is the least number for which $N_{1k} \equiv N_{2k}$.

The set of points $U = G(f=k)$ contains no point of $M_1 + M_2$. Assume that U contains no critical point of $f(x)$. We show that this assumption leads to a contradiction.

LEMMA 1. *U consists of at most a finite number of maximal connected subsets.*

PROOF. Assume the contrary; then $U = u_1 + u_2 + \cdots$, where no two of the u_i's have a point in common, each u_i is closed, and the u_i's are chosen so that the sum of two or more of them is not connected. There is at least one point p of U that is a limit point of a sequence of points $p_1, p_2, \cdots$ of U and no two of the p_j's belong to the same u_i. Since one of the partial derivatives, f_j, is different from zero at p and

f_j is continuous in R, we may find a neighborhood D of p throughout which f_j is different from zero. Furthermore, by the classical implicit function theorem* there exists a sub-neighborhood E of D such that all points of U that lie in E belong to a single continuous function $x_j = F(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)$ and hence the subset of U that lies in E is connected. This contradicts the hypothesis that p is a limit point of points of infinitely many distinct u_i's. Hence $U = U_1 + U_2 + \cdots + U_r$, where r is a positive integer and U_i, $i = 1, \cdots, r$, is closed and connected.

LEMMA 2. *U_i, $i = 1, \cdots, r$, is a closed manifold*† *in the sense of analysis situs and its connectivity R_{n-1} is given by $R_{n-1} = 2$.*

PROOF. S. S. Cairns‡ has recently shown that U_i is a complex in the sense of analysis situs. This is accomplished by breaking up U_i into $(n-1)$-cells. An application of the implicit function theorem used in proving Lemma 1 shows that U_i has each of its $(n-2)$-cells incident with exactly two $(n-1)$-cells and hence U_i has no boundary. Hence U_i is a closed manifold and since it consists of a single $(n-1)$-circuit and this is, of course, non-bounding, we have $R_{n-1} - 1 = 1$, or $R_{n-1} = 2$.

PROOF OF THEOREM 3. By Alexander's theorem§ $S - U_i$, $(i = 1, \cdots, r)$, consists of two mutually exclusive connected point sets C_i and D_i. More generally, this theorem shows that $S - \sum_{n=1}^{j} U_i$ consists of $j+1$ mutually exclusive connected sets $G_1, \cdots, G_{j+1}$. Since $S - \sum_{i=1}^{r} U_i$ consists of two mutually exclusive sets one of which contains M_1 and the other contains M_2, it follows as an immediate consequence of a theorem by Alexandroff‖ that at least one of the sets U_i, call it U_1, is such that $S - U_1 = C_1 + D_1$, where C_1 contains

* See Hobson, loc. cit., p. 410.

† See, J. W. Alexander, Transactions of this Society, vol. 23 (1922), pp. 333 ff.

‡ Unpublished work. An indication of the procedure is given by Morse, loc. cit., p. 355.

§ Loc. cit., p. 343, Theorem Y.

‖ Comptes Rendus, vol. 183, p. 723, Theorem II_n.

M_1 and D_1 contains M_2. Since C is connected, it is a subset of C_1 or D_1. A reapplication of Alexandroff's theorem yields U_2 such that $S-(U_1+U_2)=G_1+G_2+G_3$, where G_1 contains M_1, G_2 contains M_2, and G_3 contains C. For definiteness let D_1 contain M_2 and C. We distinguish between two cases: Case I, C_2 contains M_1, M_2, and U_1 while D_2 of necessity contains C; Case II, C_2 contains M_1, U_1, and C while D_2 contains M_2. No other cases can arise.

CASE 1. Since C_2 contains U_1 and since $f(x)$ is continuous over C_2+U_2, it follows that $f(x)$ assumes its maximum for this domain in at least one point of the domain. If this maximum is greater than k then the point is an interior point and a consideration of the difference quotients of which the partial derivatives of $f(x)$ are the limits shows immediately that this point is a critical point and thereby contradicts our hypothesis. If the maximum has the value k, a similar consideration of the difference quotients shows that every point of U_1 is a critical point and likewise contradicts our hypothesis.

CASE 2. In this case C_1+U_1 and D_2+U_2 are closed and mutually exclusive. $N_{1k}=[C_1+U_1]+[D_2+U_2]+Q$, where Q cannot be vacuous since N_{1k} is connected. If A is a point of Q, we find U_3 so that $S-U_3=C_3+D_3$, where C_3 contains A and D_3 contains C. If either M_1 or M_2 belongs to C_3, a repetition of the argument of Case 1 proves the existence of a critical point of the type desired for Theorem 3. Hence both C_1 and D_2 are subsets of D_3 and hence neither they nor their boundaries can contain limit points of the subset of Q that belongs to C_3. A repetition of this argument a finite number of times (at most r times) together with the observation that $\sum_{i=3}^{r} U_i$ is closed shows that $[C_1+U_1]$ and $[D_2+U_2]$ contain no limit points of Q. Hence N_{1k} cannot be connected, contrary to the manner in which it was constructed. This contradiction yields Theorem 3.

COROLLARY 1.* *If* $f(x_1, \cdots, x_n)$ *has two distinct minimal*

* This corollary follows directly from the proof of Theorem 3.

sets M_1 and M_2 in R, there exists at least one critical point of $f(x_1, \cdots, x_n)$ in R that does not belong to a minimal set of $f(x_1, \cdots, x_n)$.

COROLLARY 2. *If $f(x_1, \cdots, x_n)$ is restricted so that each of its minimal sets contains a single point, Theorem 3 becomes Bieberbach's form of the minimax principle.**

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SYMMETRIC FUNCTIONS OF n-IC RESIDUES (mod p)†

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If p be an odd prime, q is said to be an n-ic residue of p if the congruence $x^n \equiv q$ (mod p) has solutions; otherwise q is an n-ic non-residue of p. A necessary and sufficient condition that q be an n-ic residue of p is that

$$(1) \qquad q^{(p-1)/\delta} \equiv 1, \quad (\text{mod } p),$$

where $\delta = \text{g.c.d. } (p-1, n)$. The number‡ of n-ic residues of a given prime p is $(p-1)/\delta$.

It is with the symmetric functions of these n-ic residues that this paper deals.

By means of (1) we readily prove that the product of two n-ic residues is an n-ic residue and that the product of an n-ic residue by an n-ic non-residue is an n-ic non-residue.

Put $(p-1)/\delta = r$ and let $q_1, q_2, \cdots, q_r$ be the set of all distinct n-ic residues of p. Then $q_iq_1, q_iq_2, \cdots, q_iq_r$ is the same set in different order, for the assumption that two members of this last set are congruent leads to the conclusion that two members of the first set are not distinct.

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* Loc. cit., p. 140.

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‡ Dirichlet-Dedekind, *Zahlentheorie*, 4th ed., 1894, p. 74.