NON-EXISTENCE THEOREMS ON THE NUMBER OF REPRESENTATIONS OF ARBITRARY ODD INTEGERS AS SUMS OF $4r$ SQUARES*

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1. *Introduction.* The theorems stated in §3 and proved in §4 will be more significant if we first outline some known results and devise a definition which they suggest. In §5 an interesting problem is proposed, to which the method of this paper is at least partly applicable.

2. *Simplicity of the Number of Representations of an Integer as a Sum of* $4r$ *Squares.* Let n, r be given integers. The number $N(n, r)$ of one-rowed matrices $(x_1, x_2, \cdots, x_r)$ of integers $x_j \lesseqgtr 0$ $(j=1, 2, \cdots, r)$ such that $x_1^2+x_2^2+\cdots+x_r^2=n$ is called, as customary, the number of representations of n as a sum of r squares; $N(0, r)=1$. Henceforth let n be an arbitrary integer >0, and m an odd integer >0. Denote by $\zeta_j(n)$, $j\geqq 0$, the sum of the jth powers of all the divisors of n, $\zeta_j(0)=1$ by convention; and by $\xi_j(n)$ the sum of the jth powers of all the divisors $\equiv 1 \bmod 4$ of n minus the like sum for the divisors $\equiv 3 \bmod 4$. Write $(-1\,|\,m) \equiv (-1)^{(m-1)/2}$. Then, either from the analysis of Bulyguin† or otherwise, it is known that the general structure of $N(m, 2r)$ is as follows:

$$N(m,4r) = a\zeta_{2r-1}(m) + F_r(m),$$
$$N(m,4r-2) = [b + c(-1\,|\,m)]\xi_{2r-2}(m) + G_r(m),$$

where a, b, c are numerical constants (independent of m) different from zero; $F_j(m)=G_j(m)=0$ $(j=1, 2)$, and $F_r(m)$, $G_r(m)$, when $r>2$, are sums of homogeneous polynomials in the integers $y_1, y_2, \cdots, y_{2t} \gtreqless 0$ such that

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* Presented to the Society, June 20, 1929.

† Bulletin de l'Académie de St. Petersburg, 1914, pp. 389–404.

$$y_1^2+y_2^2+\cdots+y_{2t}^2=m,$$

where $t<2r$ for $F_r(m)$ and $t<2r-1$ for $G_r(m)$. For example,

$$F_6(m) = p\sum_8(y_1^8-28y_1^6y_2^2+35y_1^4y_2^4)+q\sum_{16}(y_1^4-3y_1^2y_2^2),$$

where $\sum_8$, $\sum_{16}$ refer respectively to all sets of solutions of

$$y_1^2+y_2^2+\cdots+y_8^2=m, \qquad y_1^2+y_2^2+\cdots+y_{16}^2=m,$$

respectively, and p, q are numerical constants different from zero. Precisely similar theorems hold for $N(2^\alpha m, 2r)$, $\alpha>0$; the constants a, b are then to be replaced by $a+A2^{(2r-1)\alpha}$, $b+B2^{2(r-1)\alpha}$ respectively, where A, B are numerical constants different from zero; $F_r(2^\alpha m)$, $G_r(2^\alpha m)$ replace $F_r(m)$, $G_r(m)$ and are defined in the same way as the latter with $2^\alpha m$ in place of m throughout.

We shall call $a\zeta_{2r-1}(m)$, $[b+c(-1|m)]\,\xi_{2r-2}(m)$ the *simple parts* of $N(m, 4r)$, $N(m, 4r-2)$ respectively, and similarly for $N(2^\alpha m, 2r)$. The remaining parts will be called *compound*.

DEFINITION. *If $\alpha>0$, $\beta\geqq 0$ are constant integers, and if the compound part of $N(\alpha t+\beta, 2r)$ either vanishes or is identically zero for all integers $t\geqq 0$, $N(\alpha t+\beta, 2r)$ is said to be simple.*

For example, $N(t, 2r)$ is simple when and only when $r=1, 2, 3, 4$. In these cases the compound part vanishes identically; that is, it is absent. On the other hand, $N(2t, 12)$, $N(4t+3, 10)$ are simple because the compound parts of $N(n, 12)$, $N(n, 10)$, neither of which is identically zero, vanish when $n=2t$, $n=4t+3$ respectively. These six examples exhaust the known instances of simple $N(\alpha t+\beta, 2r)$.

We shall discuss the simplicity of $N(4t+\beta, 2r)$. Evidently the following enumeration of cases is exhaustive; it is made to fit the subsequent analysis:

$$N(4t+\beta,\ 4r-j), \qquad (\beta=0,1,2,3;\ j=0,2).$$

In other papers, cited presently, all cases except $N(4t+1, 4r)$, $N(4t+3, 4r)$ have been disposed of. We may therefore restrict the discussion to these.

3. *Theorems on* $N(m, 4r)$. We shall prove the following.

THEOREM 1. *If* $N(m, 4r), m\equiv 1 \bmod 4$, *is simple*, $N(m, 4r) = 8r\zeta_{2r-1}(m)$.

THEOREM 2. *If* $N(m, 4r), m\equiv 3 \bmod 4$, *is simple*,

$$N(m,4\ ) = \frac{32r(2r-1)(4r-1)}{3(3^{2r-1}+1)}\zeta_{2r-1}(m).$$

THEOREM 3. *If* $m\equiv 1 \bmod 4$, *and* $N(m, 4r+4)$, $r>1$, *is simple, it is necessary* (*but not sufficient*) *that*

$$3\cdot 5^{2r+2} = 512r^4 + 768r^3 + 352r^2 + 168r + 225.$$

THEOREM 4. *If* $m\equiv 3 \bmod 4$, *and* $N(m, 4r+4)$, $r>2$, *is simple, it is necessary* (*but not sufficient*) *that*

$$105(7^{2r+1}+1) = (512r^4 - 256r^3 - 32r^2 + 856r + 210)(3^{2r+1}+1).$$

THEOREM 5. $N(m, 4r+4)$ *is simple for no* $r>1$.

THEOREM 6. $N(m, 4r)$ *is simple when and only when* $r=1, 2$.

To state the next theorem, let k be the modulus of the elliptic function sn u. Then, as is well known or easily seen by Maclaurin's theorem, the coefficient of $(-1)^r u^{2r+1}/(2r+1)!$ $(r\geqq 0)$ in the expansion of sn u is

$$S_{2r+1}(k^2) \equiv \sum_{t=0}^{r} s_t(r)k^{2t},$$

where the $s_t(r)$ $(t=0, 1, \cdots, r)$ are integers >0. The expansion being unique, the integers $s_t(r)$ are uniquely defined. Let $\binom{p}{j}$ denote the coefficient of x^j in the expansion of $(1+x)^p$ if $p>0$ and $0\leqq j\leqq p$, and zero otherwise. Then

$$\sigma_\mu(r) \equiv \sum_{t=0}^{\mu} 2^{2t}\binom{2r+1-2t}{\mu-t}s_t(r)$$

is an integer >0 whenever $r\geqq 0$, $\mu\geqq 0$ are integers.

THEOREM 7. *If* $m\equiv 1 \bmod 4$, *a necessary and sufficient condition that* $N(m, 4r+4)$, $r\geqq 0$, *shall be simple, is that a constant* λ (*independent of* m) *shall exist such that*

$$\sigma_{2\mu}(r) = \lambda\binom{4r+4}{4\mu+1}, \qquad (\mu = 0, 1, \cdots, r);$$

if such a λ *exists*, $\lambda N(m, 4r) = 2\zeta_{2r+1}(m)$.

THEOREM 8. *If* $m \equiv 3 \bmod 4$, *a necessary and sufficient condition that* $N(m, 4r+4)$, $r \geqq 0$, *shall be simple, is that a constant* λ (*independent of* m) *shall exist such that*

$$\sigma_{2\mu+1}(r) = \lambda\binom{4r+4}{4\mu+3}, \qquad (\mu = 0, 1, \cdots, r);$$

if such a λ *exists*, $N(m, 4r+4) = 2\zeta_{2r+1}(m)$.

4. *Proofs*. It is readily seen that Theorems 7, 8 imply all the rest. We shall require the following explicit values of $s_t(r)$ ($t = 0, 1, 2, 3$) for all integers $r \geqq 0$, due to Hermite:*

$$\begin{aligned} s_0(r) &= 1\ ;\ 2^4 s\ (r) = 3^{2r+1} - 8r - 3\ ; \\ 2^8 s_2(r) &= 5^{2r+1} - (8r-4)3^{2r+1} + 32r^2 - 32r - 17\ ; \\ 2^{12} s_3(r) &= 7^{2r+1} - (8r-12)5^{2r+1} + (32r^2 - 88r + 30)3^{2r+1} \\ &\quad - \tfrac{1}{3}(256r^3 - 1056r^2 + 752r + 471). \end{aligned}$$

Assume for a moment that Theorems 7, 8 are true. Then the value of λ is found by taking $\mu = 0$. Theorems 1, 2 result. Take $\mu = 0, 1$ in Theorems 7, 8; Theorems 3, 4 then follow by a simple but rather tedious calculation in an obvious manner. Theorem 5 follows from Theorems 3, 4 on remarking that for the appropriate (small) value of $r > 2$ the left-hand members of the equalities stated in Theorems 3, 4 become and remain greater than the respective right-hand members; the few values of r not thus rejected are thrown out by Theorems 7, 8. (The details of a similar calculation are given in the paper cited presently.) Theorem 6 is then immediate. It remains then only to prove Theorems 7, 8.

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* Oeuvres, vol. 3, p. 237; stated without indication of proof. As certain others of Hermite's values contain misprints, the above were calculated by the method of Gruder, Wiener Sitzungsberichte, vol. 126, IIa (1917), and found correct.

It was proved in a former paper* that

$$4 \sum_m q^{m/2}\zeta_{2r+1}(m) = \sum_{t=0}^{r} s_t(r)\theta_2^{4\,t+2}\theta_3^{4\,r+2-4t}, \quad |q| < 1,$$

for all integers $r \geqq 0$, where $\sum_m$ refers to all odd integers $m > 0$, and θ_2, θ_3 are the usual elliptic theta constants. It was also pıoved that for each value of the integer $r \geqq 0$, the set of integer coefficients $s_t(r)$ $(t=0, 1, \cdots, r)$ can not be replaced by any other set of $r+1$ integers >0. Apply to the above the identities

$$\theta_2^2(q^{1/2}) = 2\theta_2\theta_3, \qquad \theta_3^2(q^{1/2}) = \theta_2^2 + \theta_3^2$$

from the transformation of the second order. A short reduction gives

$$2 \sum_m q^{m/4}\zeta_{2r+1}(m) = \sum_{\mu=0}^{2r+1} \sigma_\mu(r)\theta_2^{2\mu+1}\theta_3^{4\,r+3-2\mu}.$$

From the uniqueness of the set of coefficients $s_t(r)(t=0, 1, \cdots, r)$ and the manner in which the last identity has been obtained, it follows by a simple contradiction that the $\sigma_\mu(r)$ $(\mu=0, 1, \cdots, 2r+1)$ can not be replaced by any other set of integers >0. Replace q by q^4, and denote by $N(n, r, h)$ the number of representations of n as a sum of r squares, precisely h of which are odd. Since a sum of r squares, precisely h of which are odd, is $\equiv h$ mod 4, we compare coefficients of like powers of q and separate cases modulo 4. Thus

$$m \equiv 1 \bmod 4:\ 2\zeta_{2r+1}(m) = \sum_{\mu=0}^{r} \sigma_{2\mu}(r)\frac{N(m, 4r+4, 4\mu+1)}{\binom{4r+4}{4\mu+3}};$$

$$m \equiv 3 \bmod 4:\ 2\zeta_{2r+1}(m) = \sum_{\mu=0}^{r} \sigma_{2\mu+1}(r)\frac{N(m, 4r+4, 4\mu+3)}{\binom{4r+4}{4\mu+3}}.$$

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* To appear shortly in the Journal of the London Mathematical Society. The paper dealing with the remaining cases of $N(4t+\beta, 2r)$ has not yet been published.

By what precedes, the $\sigma_j(r)$ can not be replaced by any other sets of integers >0. But, obviously,

$$m \equiv 1 \bmod 4: \; N(m, 4r+4) = \sum_{\mu=0}^{r} N(m, 4r+4, 4\mu+1)\,;$$

$$m \equiv 3 \bmod 4: \; N(m, 4r+4) = \sum_{\mu=0}^{r} N(m, 4r+4, 4\mu+3)\,.$$

Hence Theorems 7, 8 follow.

5. *Statement of a General Problem.* The theorem that $N(t,\ 2r)$ is simple when $r=1,\ 2,\ 3,\ 4$ is due to Eisenstein.* That $N(t,\ 2r)$ is simple only for those values of r does not seem to follow from his statements; the simplicity when and only when $r=1,\ 2,\ 3,\ 4$ can be proved readily by the methods of this paper. It follows from the theorems of the present paper and the others cited in §4 that $N(t,\ 2r)$, $N(4t+\beta,\ 2r)$ are simple only for the values of β, r stated in §1. Further, it is clear that the complete theory of the simplicity of $N(\alpha t+\beta,\ 2r)$ is known for $\alpha=1,\ 2,\ 4$. This suggests the following problem. *Find all values of r (if any) for which $N(\alpha t+\beta,\ 2r)$ is simple, where α is an arbitrary integer >0 and different from* 1, 2, 4. The obvious generalization in which the ζ, ξ functions are replaced by other given functions of divisors might also be considered.

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* Journal für Mathematik, vol. 35 (1847), p. 135.