NOTE ON LINEAR TRANSFORMATIONS OF n-ICS IN m VARIABLES*

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Let us consider the n-ic in m variables

$$F(x_1, x_2, \cdots, x_m) = 0. \tag{1}$$

If we subject (1) to the linear transformation

$$\begin{aligned} \rho x_1 &= a_{11}x_1' + a_{12}x_2' + a_{13}x_3' + \cdots + a_{1m}x_m', \\ \rho x_2 &= a_{21}x_1' + a_{22}x_2' + \cdots + a_{2m}x_m', \cdots, \\ \rho x_m &= a_{m1}x_1' + a_{m2}x_2' + a_{m3}x_3' + \cdots + a_{mm}x_m', \end{aligned} \tag{2}$$

we obtain

$$\begin{aligned} &F(a_{11}x_1' + a_{12}x_2' + a_{13}x_3' + \cdots + a_{1m}x_m', a_{21}x_1' + a_{22}x_2' \\ &\quad + a_{23}x_3' + \cdots + a_{2m}x_m', \cdots, a_{m1}x_1' + a_{m2}x_2' \\ &\quad + a_{m3}x_3' + \cdots + a_{mm}x_m') = 0. \end{aligned} \tag{3}$$

Note that in the expansion of (3) the coefficient of the term in $x_i'^n$, $(i=1, 2, 3, \cdots, m)$, is $F(a_{1i}, a_{2i}, a_{3i}, \cdots, a_{mi})$. A necessary and sufficient condition for this coefficient to vanish is that the point $P_i(a_{1i}, a_{2i}, \cdots, a_{mi})$ shall lie on the geometric locus of (1). To obtain the coefficient of such a term as $x_i'^r x_j'^{n-r}$ in the expansion of (3) we can put

$$\begin{aligned} &x_i' x_j' \neq 0,\ x_1' = x_2' = x_3' = \cdots = x_{i-1}' \\ &\quad = x_{i+1}' = x_{i+2}' = \cdots = x_{j-1}' = x_{j+1}' = \cdots = x_m' = 0, \end{aligned}$$

then use Taylor's Expansion on

$$\begin{aligned} &F(a_{1i}x_i' + a_{1j}x_j', a_{2i}x_i' + a_{2j}x_j', a_{3i}x_i' \\ &\quad + a_{3j}x_j', \cdots, a_{mi}x_i' + a_{mj}x_j') \equiv F(X_1 + X_1', \\ &X_2 + X_2', X_3 + X_3', \cdots, X_m + X_m'), \end{aligned} \tag{4}$$

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where $X_1 = a_{1i}x_i'$, $X_1' = a_{1j}x_j'$, $X_2 = a_{2i}x_i'$, $X_2' = a_{2j}x_j'$, etc. We find the coefficient of $x_i'^{\,r}x_j'^{\,n-r}$ in the group of terms*

$$\frac{1}{(n-r)!}\left(\frac{\partial F}{\partial X_1}X_1' + \frac{\partial F}{\partial X_2}X_2' + \frac{\partial F}{\partial X_3}X_3' + \cdots + \frac{\partial F}{\partial X_m}X_m'\right)^{(n-r)}$$

$$(5)\qquad \equiv \frac{(x_i'^{\,r}x_j'^{\,n-r})}{(n-r)!}\left(\frac{\partial F}{\partial a_{1i}}a_{1j} + \frac{\partial F}{\partial a_{2i}}a_{2j} + \frac{\partial F}{\partial a_{3i}}a_{3j} + \cdots + \frac{\partial F}{\partial a_{mi}}a_{mj}\right)^{(n-r)},$$

where $\partial F/\partial a_{1i}$ means $\partial F/\partial X_1$ with X_1 replaced by a_{1i}, X_2 by a_{2i}, X_3 by $a_{3i}, \cdots, X_m$ by a_{mi}, and similarly for $\partial F/\partial a_{2i}$, $\partial F/\partial a_{3i}, \cdots, \partial F/\partial a_{mi}$. Hence we may conclude that a necessary and sufficient condition for the vanishing of the coefficient of $x_i'^{\,r}x_j'^{\,n-r}$ in the expansion of (3) is that the point $P_j(a_{1j}, a_{2j}, a_{3j}, \cdots, a_{mj})$ shall lie on the $(n-r)$th polar of $P_i(a_{1i}, a_{2i}, \cdots, a_{mi})$ with respect to the locus of (1).

We obtain the coefficient of the term in $x_i'^{\,r}x_j'^{\,s}x_k'^{\,t}\cdots x_l'^{\,u}x_p'^{\,v}$ (where $r+s+t+\cdots+u+v=n$) in the expansion of (3) by the following device. We can write (3) as

$$(6)\quad F(Y_1 + Y_1', Y_2 + Y_2', Y_3 + Y_3', \cdots, Y_m + Y_m') = 0,$$

where

$$Y_1 = a_{1i}x_i', Y_1' = a_{11}x_1' + a_{12}x_2' + a_{13}x_3' + \cdots + a_{1i-1}x_{i-1}' + a_{1i+1}x_{i+1}' + a_{1i+2}x_{i+2}' + \cdots + a_{1m}x_m',$$

$$Y_2 = a_{2i}x_i', Y_2' = a_{21}x_1' + a_{22}x_2' + \cdots + a_{2i-1}x_{i-1}' + a_{2i+1}x_{i+1}' + \cdots + a_{2m}x_m',$$

etc. We take the collection of terms

$$(7)\qquad \frac{1}{(n-r)!}\left(\frac{\partial F}{\partial Y_1}Y_1' + \frac{\partial F}{\partial Y_2}Y_2' + \frac{\partial F}{\partial Y_3}Y_3' + \cdots + \frac{\partial F}{\partial Y_m}Y_m'\right)^{(n-r)}$$

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* See Goursat-Hedrick *Mathematical Analysis*, vol. 1, pp. 107–108. For the Galois fields, see A. D. Campbell, *The polar curves of plane algebraic curves in the Galois fields*, this Bulletin, vol. 34 (1928), pp. 361–363. The methods of this paper may be readily generalized to the polars of an n-ic in m variables.

in the expansion of (6). All the terms with $x_i'^r$ as a factor must come from (7). We can write (7) in the form

$$(8)\qquad \frac{x_i'^r}{(n-r)!}\left(\frac{\partial F}{\partial a_{1i}}Y_1' + \frac{\partial F}{\partial a_{2i}}Y_2' + \frac{\partial F}{\partial a_{3i}}Y_3' + \cdots + \frac{\partial F}{\partial a_{mi}}Y_m'\right)^{(n-r)}.$$

If we equate (8) to zero we obtain the $(n-r)$th polar of $P_i(a_{1i}, a_{2i}, a_{3i}, \cdots, a_{mi})$ with respect to the locus of (1). We can also write (7) in the form

$$(9)\qquad \frac{x_i'^r}{(n-r)!}F'(Y_1', Y_2', Y_3', \cdots, Y_m'),$$

where F' is a function of the $(n-r)$th degree. We put

$$(10)\ Y_1' = Z_1 + Z_1', Y_2' = Z_2 + Z_2', \cdots, Y_m' = Z_m + Z_m',$$

where

$$Z_1 = a_{1j}x_j', Z_1' = a_{11}x_1' + a_{12}x_2' + a_{13}x_3' + \cdots + a_{1i-1}x_{i-1}' + a_{1i+1}x_{i+1}' + a_{1i+2}x_{i+2}' + \cdots + a_{1j-1}x_{j-1}' + a_{1j+1}x_{j+1}' + a_{1j+2}x_{j+2}' + \cdots + a_{1m}x_m', Z_2 = a_{2j}x_j',$$

etc. Expanding (9), we find that all the terms in the expansion of (3) that have the factors $x_i'^r$ and $x_j'^s$ must be in the collection of terms

$$(11)\qquad \frac{x_i'^r}{(n-r)!(n-r-s)!}\left(\frac{\partial F'}{\partial Z_1}Z_1' + \frac{\partial F'}{\partial Z_2}Z_2' + \frac{\partial F'}{\partial Z_3}Z_3' + \cdots + \frac{\partial F'}{\partial Z_m}Z_m'\right)^{(n-r-s)} \equiv \frac{(x_i'^r x_j'^s)}{(n-r)!(n-r-s)!}\left(\frac{\partial F'}{\partial a_{1j}}Z_1' + \frac{\partial F'}{\partial a_{2i}}Z_2' + \frac{\partial F'}{\partial a_{3j}}Z_3' + \cdots + \frac{\partial F'}{\partial a_{mj}}Z_m'\right)^{(n-r-s)}.$$

If we equate (11) to zero we shall have the $(n-r-s)$th polar of the point $P_j(a_{1j}, a_{2j}, a_{3j}, \cdots, a_{mj})$ with respect to the $(n-r)$th polar of $P_i(a_{1i}, a_{2i}, a_{3i}, \cdots, a_{mi})$ with respect to the locus of (1).

Next we take

$$Z_1' = W_1 + W_1', \; Z_2' = W_2 + W_2',$$
$$Z_3' = W_3 + W_3', \cdots, Z_m' = W_m + W'_m$$

in (11), where

$$W_1 = a_{1k}x_k', \; W_1' = a_{11}x_1' + a_{12}x_2' + \cdots + a_{1i-1}x_{i-1}'$$
$$+ a_{1i+1}x_{i+1}' + \cdots + a_{1j-1}x_{j-1}' + a_{1j+1}x_{j+1}' + \cdots$$
$$+ a_{1k-1}x_{k-1}' + a_{1k+1}x_{k+1}' + \cdots + a_{1m}x_m', \; W_2 = a_{2k}x_k',$$

etc. We repeat the above processes until we finally reach the collection of terms having all the factors $x_i'^r$, $x_j'^s$, $x_k'^t, \cdots, x_l'^u$, and $x_p'^v$. Therefore, we see that for the coefficient of the term $x_i'^r x_j'^s x_k'^t \cdots x_l'^u x_p'^v$ in the expansion of (3) to vanish we must have the point $P_p(a_{1p}, a_{2p}, a_{3p}, \cdots, a_{mp})$ on the $(n-r-s-t- \cdots -u-v)$th polar of $P_l(a_{1l}, a_{2l}, \cdots, a_{ml})$ with respect to the $\cdots$ th polar of $\cdots, \cdots, \cdots$, with respect to the $(n-r-s-t)$th polar of $P_k(a_{1k}, a_{2k}, \cdots, a_{mk})$ with respect to the $(n-r-s)$th polar of $P_j(a_{1j}, a_{2j}, \cdots, a_{mj})$ with respect to the $(n-r)$th polar of $P_i(a_{1i}, a_{2i}, \cdots, a_{mi})$ with respect to the locus of (1).

It is noteworthy that this discussion applies to the ordinary complex or real domains and also to the Galois fields.

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