FRAENKEL ON GRUNDLEGUNG DER MENGENLEHRE

Zehn Vorlesungen über die Grundlegung der Mengenlehre. (Wissenschaft und Hypothese, XXXI). By Adolph Fraenkel. Leipzig and Berlin, B. G. Teubner, 1927. x+182 pp.

This book embodies a series of ten lectures delivered in Kiel, June 8–12, 1925, at the invitation of the *Kant-Gesellschaft, Ortsgruppe Kiel.* The objects to be accomplished by its publication seem to be two-fold: (1) To render intelligible to non-workers in the field, especially philosophers, past and present views on the "Foundations," and (2) to stimulate reflection amongst the mathematical public concerning the foundations of mathematical thought and methods.

With regard to (1) it seems to the reviewer that the author has been singularly successful. We have here in no sense a formal text, the informal nature of the lectures having been preserved, with desirable additions made as a result of the subsequent discussion, together with literary references, etc. And yet the treatment is not superficial. The author plunges at once into a discussion of the notions of "set", equivalence of sets and cardinal numbers, order types, the continuum problem, the "diagonal procedure", antinomies and paradoxes. Of the last are given the (Zermelo-) Russell Paradox, the Burali-Forti Antinomy and Richards' Paradox. The consequences of the antinomies in mathematical thought are briefly mentioned—the failure to give a definition of the "set" notion less "offensive" than the Cantorean, in whose looseness seems to lie the germ of such paradoxes as Russell's—the suggestion that the fault be ascribed to logic, not to mathematics—the resignation of such men as Dedekind and Frege to what they considered the inevitable results of the antinomies.

The remainder of the lectures is devoted to the efforts of various schools so to revise the guiding rules of mathematical technique that the antinomies cannot arise. An exposition is given of the views of the Brouwer school (Intuitionism), as well as of the modified intuitionism of Poincaré, and the changes in the structure of the mathematical edifice that follow an adoption of the intuitionist views. There are also references to the system of Russell and Whitehead.

By far the greater part of the book (from page 58 on), however, is given over to the axiomatic foundation of the Mengenlehre begun by Zermelo. A pleasing feature of this part of the work is the way in which the author leads up to outstanding unsolved problems—certainly the mathematician who reads will *feel* the urge to apply himself to their solution, whether or not he actually yields to the temptation. Another pleasing feature is the occasional discussion of much debated matters, such as the "choice axiom", the non-predicative procedure and the excluded third. The axioms are seven in number. One of these is the choice axiom. Another axiom definitely implies the use of the logical law of the excluded third. And the non-predicative method seems to be indispensable for the axiomatic development.

The author's justification of these inclusions is exceedingly interesting from any point of view, be it philosophical, mathematical or purely logical. Thus, in regard to the choice axiom, after a historical summary of its use, its formulation, and criticisms directed against it, the author maintains "This logical principle has certainly at least a character of evidence and necessity equivalent to that which one attributes to many other axioms that one is accustomed to recognize as indispensable for the foundations of arithmetic, analysis and geometry; it was approved even by one so close to the intuitionist viewpoint as Poincaré. With equal right, then, to that with which one rejects the choice axiom, one could arbitrarily deny other important fundamental principles, and so in the future ban important parts of mathematics." The placing of the choice principle in the axioms is then compared to the assumption of the parallel axioms by the Greeks. This analogy leads the reviewer to wonder if the denial of the choice principle might not lead to fruitful results as did the denial of the parallel axiom. A non-euclidean geometry proved to be a happy conception, and might there not possibly be some significance in a non-Zermeloan Mengenlehre which asserts the existence of sets that cannot be well-ordered?* On a historical basis, at least, one would seem justified in maintaining that the denial of "important fundamental principles" is not necessarily an iconoclasm.

What would probably comprise the tenth lecture (the division of topics occurs in pairs of lectures) is devoted to a discussion of consistence, independence and categoricalness of axiomatic systems in general as well as of the particular system presented for the Mengenlehre. Although the system which Fraenkel gives does away with the existing paradoxes and antinomies, its consistency is an open question. From the point of view of him who accepts the axiomatic foundation this is a great defect—indeed, it can be said that if ever a consistency proof were needed, it is needed in this particular instance, since the very purpose of the adoption of the axiomatic method in this case is to avoid contradiction.

The reviewer heartily recommends these lectures to any who wish a general summary of the state of the "Foundations," as well as to those who wish a good introduction for mature students. The book is finished off with an unusually complete bibliography comprising eight pages, and this is amplified by up-to-date additions made during the period of proof-reading.

R. L. Wilder

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* See Church, *Alternatives to Zermelo's assumption*, Transactions of this Society, vol. 29 (1927), p. 178.