MODERN HYDRODYNAMICAL THEORY, WITH SPECIAL REFERENCE TO AERONAUTICS*

BY F. D. MURNAGHAN

A. GENERAL NOTIONS AND THEOREMS

1. *Particle and Local Differentiation.* In hydrodynamical theory matter is regarded as distributed continuously, but its various points are supposed to be capable of identification; we shall refer to them as particles. In order to identify a particle, a material reference frame is introduced, and we shall suppose for simplicity that this is a rectangular Cartesian frame. At any convenient initial instant $t = t_0$, the coordinates of a particle may be denoted by (a, b, c); and at any later instant the coordinates of the same particle may be denoted by (x, y, z). We make the assumption that (x, y, z) are differentiable functions of a, b, c, and t. This assumption
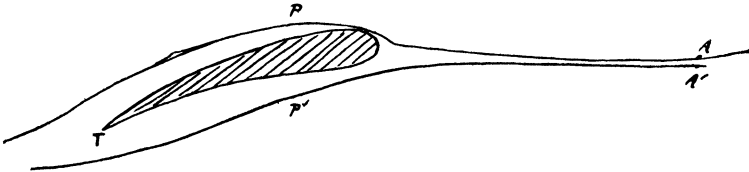

FIG. 1

must be carefully noted since the principal object of interest to us later on will be the flow of a fluid around an obstacle and the fluid will divide at the obstacle. Thus two particles initially adjacent (as A and A' in Fig. 1) will not be adjacent when separated by the obstacle as at P and P'. Our subse-

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quent remarks will apply, then, separately to the two por-
tions of the fluid which separate at the obstacle but not to
the fluid as a whole. A particle may be identified either by
means of (a, b, c, t) which are known as the particle, or La-
grangian, variables or by means of (x, y, z, t) which are
known as the local, or Eulerian, variables; we shall denote
time differentiation when the particle variables are used by
d/dt and when the local variables are used by $\partial/\partial t$. The
velocity components $(dx/dt, dy/dt, dz/dt)$ being denoted by
(u, v, w), we have the relation

$$(1) \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \cdot$$

In particular the acceleration components $(du/dt, dv/dt,$
$dw/dt)$ are given, in terms of the local variables, by formulas
of the type

$$(2) \qquad \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \cdot$$

The paths of the various particles are found, if (u, v, w) are
known in terms of the local variables, by integrating the
three equations

$$(3) \qquad \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = dt.$$

At any given instant $t = t_1$ the *stream-lines* are given by inte-
grating the *two* equations

$$(4) \qquad \frac{\delta x}{u_1} = \frac{\delta y}{v_1} = \frac{\delta z}{w_1},$$

where $u_1 = u(x, y, z, t_1)$, and so on. The ∞^3 paths must then
be distinguished from the ∞^2 stream-lines. They coincide
geometrically when the stream-lines are permanent, that is,
when $\partial u/\partial t = 0$, $\partial v/\partial t = 0$, $\partial w/\partial t = 0$ identically in t. In this
case the motion is said to be *steady*, and in the applications
with which we shall be concerned steady motion is assumed.

2. *The Principle of Conservation of Mass.* This postulates the existence of a positive density function ρ, which we assume to be differentiable, and such that the volume integral $\int\rho\ \delta(x,\ y,\ z)$ does not vary with t, the volume of integration consisting of the same particles. In other words $\int\rho\ \delta(x,\ y,\ z)$ is an integral invariant of the equations (3). If the initial value of ρ be denoted by ρ_0, the differential form of statement of the principle of conservation is, in terms of the particle variables,

$$\rho\frac{d(x,y,z)}{d(a,b,c)} = \rho_0,$$

and in terms of the local variables,

$$(5) \qquad \frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0.$$

This equation is known as the equation of continuity and it may be put in the equivalent form

$$\frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) = 0.$$

For an incompressible fluid $d\rho/dt=0$ and the equation of continuity is

$$(6) \qquad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0.$$

3. *The Circulation Integral.* With any closed curve C of particles we may associate, at any instant t, the number

$$(7) \qquad \Gamma = \oint (u\,\delta x + v\,\delta y + w\,\delta z),$$

where δ indicates the differential with respect to the parameter of the curve. The integral Γ is known as the circulation around the curve; and an application of Stokes' theorem enables us to write it in the equivalent form

(8) $$\Gamma = \int \xi\, \delta(y,z) + \eta\, \delta(z,x) + \zeta\, \delta(x,y),$$

where

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

are the vorticity components and the surface of integration is bounded by C. A simple calculation shows that

(9)
$$\frac{d\Gamma}{dt} = \oint \left\{ \left(\frac{\partial}{\partial y}\frac{dw}{dt} - \frac{\partial}{\partial z}\frac{dv}{dt} \right) \delta(y,z) \right.$$

$$+ \left(\frac{\partial}{\partial z}\frac{du}{dt} - \frac{\partial}{\partial x}\frac{dw}{dt} \right) \delta(z,x) + \left(\frac{\partial}{\partial x}\frac{dv}{dt} - \frac{\partial}{\partial y}\frac{du}{dt} \right) \delta(x,y) \left. \right\}.$$

Hence the necessary and sufficient conditions that $d\Gamma/dt = 0$ for every closed curve C are

$$\frac{\partial}{\partial y}\frac{dw}{dt} - \frac{\partial}{\partial z}\frac{dv}{dt} = 0,$$

and so forth. In other words, Γ is a (relative) integral invariant of the equations (3) provided that the acceleration components are derivable from a potential function Q, that is, provided that

(10) $$\frac{du}{dt} = -\frac{\partial Q}{\partial x}, \quad \frac{dv}{dt} = -\frac{\partial Q}{\partial y}, \quad \frac{dw}{dt} = -\frac{\partial Q}{\partial z},$$

and conversely. In our applications this will be the situation. A particular instance of importance occurs when $\Gamma = 0$ throughout a region of the fluid; then the vorticity components $(\xi,\ \eta,\ \zeta)$ are zero and the motion is said to be *irrotational* in that region. The velocity components are derivable from a single function ϕ as follows:

(11) $$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial z};$$

and ϕ is called the velocity potential.

4. *The Fundamental Equations and the Concept of a Perfect Fluid.* If we focus our attention on a particular point (x, y, z) of the fluid, the orientation of a particular element of area δS through (x, y, z) may be specified by means of the direction cosines (l, m, n) of one of the two directed normals to δS and we may call the side of δS towards which the normal is drawn its positive side. Then the fluid on the negative side of δS exerts on the fluid on the positive side a force whose components may be denoted by $(P_x \delta S, P_y \delta S, P_z \delta S)$, where (P_x, P_y, P_z) are given in terms of (l, m, n) and nine coefficients $(p_{xx}, p_{xy}, \cdots, p_{zz})$ by the equations

$$
\begin{aligned}
P_x &= l p_{xx} + m p_{yx} + n p_{zx}, \\
(12) \qquad P_y &= l p_{xy} + m p_{yy} + n p_{zy}, \\
P_z &= l p_{xz} + m p_{yz} + n p_{zz}.
\end{aligned}
$$

The nine coefficients $(p_{xx}, p_{xy}, \cdots, p_{zz})$ constitute what is called the stress or pressure tensor. If the fluid is subjected to a system of forces (X, Y, Z) per unit mass, a simple application of the fact that the total force acting on any portion of the fluid is equal to the time rate of change of its linear momentum yields the equations

$$
(13) \qquad \rho \frac{du}{dt} = \rho X - \left(\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z} \right) ;
$$

$$
\rho \frac{dv}{dt} = \qquad ; \quad \rho \frac{dw}{dt} = \qquad \cdot
$$

In deriving these we have used the fact that if there are surface forces, whose components are (F, G, H) per unit area, acting on the fluid then (see (1)) we have

$$
(14) \quad F + l p_{xx} + m p_{yx} + n p_{zz} = 0 \; ; \; G + \quad = 0 \; ; \; H + \quad = 0 \; ;
$$

where (l, m, n) are the direction cosines of the outward drawn normal. The theorem that the moment of the forces acting on the fluid about any axis is equal to the time rate of change of its angular momentum about this axis yields the fact that the pressure tensor is symmetric, that is,

(15) $p_{zy} = p_{yz}, \quad p_{xz} = p_{zx}, \quad p_{yx} = p_{xy}.$

A perfect fluid is by definition one such that the stress across any element is normal to it; it is this quality of inability to withstand tangential stresses that distinguishes fluids from solids. The equations (12) combined with $P_x:P_y:P_z = l:m:n$ for arbitrary values of the ratios $l:m:n$ show that the pressure tensor for a perfect fluid is the *scalar* tensor $(p, 0, 0; 0, p, 0; 0, 0, p)$, where p is the common value of p_{xx}, p_{yy}, p_{zz}. The number p is called the *pressure* of the perfect fluid and it is well to remember that the concept of pressure, as here defined, does not exist for a non-perfect or *viscous* fluid. For a perfect fluid the equations of motion are

(16) $\rho\dfrac{du}{dt} = \rho X - \dfrac{\partial p}{\partial x}, \quad \rho\dfrac{dv}{dt} = \rho Y - \dfrac{\partial p}{\partial y}, \quad \rho\dfrac{dw}{dt} = \rho Z - \dfrac{\partial p}{\partial z},$

to which must be added the boundary conditions

(17) $F + lp = 0, \quad G + mp = 0, \quad H + np = 0,$

where p denotes, in these latter equations, the pressure at the boundary of the fluid. It is clear from (16) that an acceleration potential Q exists for a perfect fluid if (1) (X, Y, Z) are derivable from a potential function V, that is, if

$$X = -\frac{\partial V}{\partial x}, \quad Y = -\frac{\partial V}{\partial y}, \quad Z = -\frac{\partial V}{\partial z},$$

and (2) if

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial P}{\partial x}, \quad \frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{\partial P}{\partial y}, \quad \frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{\partial P}{\partial z},$$

where P is any function of x, y, z and t. Then $Q = V+P$. We shall be concerned mainly with the case where the fluid is *homogeneous*, that is, $\partial\rho/\partial x = 0$, $\partial\rho/\partial y = 0$, $\partial\rho/\partial z = 0$, and in this case $P = p/\rho$ so that

(18) $Q = V + \dfrac{p}{\rho}.$

If our first condition is satisfied but not our second we have from (9)

$$\frac{d\Gamma}{dt} = \int \left[\frac{\partial\left(p,\dfrac{1}{\rho}\right)}{\partial(y,z)} \delta(y,z) + \frac{\partial\left(p,\dfrac{1}{\rho}\right)}{\partial(z,x)} \delta(z,\,x) \right.$$

$$\left. + \frac{\partial\left(p,\dfrac{1}{\rho}\right)}{\partial(x,y)} \delta(x\,,y) \right]$$

(19)

$$= \int \delta\left(p,\frac{1}{\rho}\right) = \oint_c p\delta\left(\frac{1}{\rho}\right).$$

5. *Bernoulli's Theorem connecting Pressure and Velocity.* When an acceleration potential Q exists the three equations $du/dt = -\partial Q/\partial x$, etc., may be written in the form

$$\frac{\partial u}{\partial t} + \eta w - \zeta v = -\frac{\partial}{\partial x}\left(Q + \frac{1}{2}q^2\right),$$

and so forth, where $(\xi,\ \eta,\ \zeta)$ are the vorticity components and $q^2 = u^2 + v^2 + w^2$ is the square of the velocity. These yield at once the equation

$$\frac{d}{dt}\left(Q + \frac{1}{2}q^2\right) = \frac{\partial Q}{\partial t},$$

so that when Q is free of t, $Q + q^2/2$ is a first integral of the equations (3). In other words $Q + q^2/2$ has a constant value along any one path but it will, in general, vary from path to path. If, however, the motion is *steady* ($\partial u/\partial t = 0$, etc.), and *irrotational* ($\xi = 0$, etc.), we have $\partial(Q + q^2/2)/\partial x = 0$, etc., so that $Q + q^2/2$, being free of t, is a number independent of x, y, z and t. In the application with which we shall be concerned we shall have $V = 0$, $P = p/\rho$, so that $Q = V + P = p/\rho$, with ρ a numerical constant. Hence we may write

(20) $p + \frac{1}{2}\rho q^2 = p_0$;   or $p = p_0 - \frac{1}{2}\rho q^2$,

where p_0 is a constant throughout the fluid. Since p_0 is

what p becomes wherever $q = 0$, it is called the *static* pressure. The additional term $-\rho q^2/2$ is called the *dynamic* pressure.

6. *The Forces and Turning Moments Experienced by Solids in the Fluid.* The equations (16) may be written in the form

$$(21) \quad \rho X - \frac{\partial p}{\partial x} = \rho \frac{du}{dt} = \rho \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho uv) + \frac{\partial}{\partial z}(\rho uw),$$

and so forth, on using (5). Upon integrating this equation throughout the region of the fluid between the immersed solid, or solids, S and any convenient outer boundary surface S', and using the fact that at any point of the surface of a solid the velocity of the fluid is tangential to the surface, we find that the force components (F_x, F_y, F_z) exerted by the fluid on the immersed solids are given by equations of the form

$$(22) \quad F_x = \int \rho X d\tau - \int \frac{\partial}{\partial t}(\rho u) d\tau$$

$$- \int \rho u(lu + mv + nw)dS' - \int p l dS',$$

where (l, m, n) are the direction cosines of the normal to dS', drawn away from the fluid. In the cases to which we shall apply this result we shall assume the applied mass forces negligible ($X = 0$, etc.), the motion steady ($\partial u/\partial t = 0$, etc.) and irrotational (so that (20) is applicable) and the fluid is compressible and homogeneous; we find then

$$(23) \quad F_x = \frac{\rho}{2} \int \left\{ l(u^2 + v^2 + w^2) - 2u(lu + mv + nw) \right\} dS';$$

$$F_y = \qquad ; \quad F_z = \qquad .$$

In a similar way, the moments (M_x, M_y, M_z) about the co-ordinate axes of the forces exerted by the fluid on the immersed obstacle are given by equations of the form

$$(24) \quad M_x = \frac{\rho}{2} \int \left\{ (yn - zm)(u^2 + v^2 + w^2) \right.$$

$$\left. - 2(yw - zv)(lu + mv + nw) \right\} dS',$$

and so forth. It is with these forces and moments that we shall be mainly concerned and the significance of the formulas given lies in the fact that the desired values are obtained by integration over an *arbitrary* surface S' enclosing the obstacles. It will be usually convenient, when the fluid is supposed infinite in extent, to regard S' as receding to infinity and we see that a knowledge of the nature of the flow at infinity suffices to determine the forces exerted on the solids by the fluid and also the moments of these forces.

B. Two-Dimensional Irrotational Motion

7. *The Complex Velocity and Potential.* In many applications we may assume, at any rate to a first approximation, that the velocity of each particle is parallel to a given plane and independent of its distance from it. Taking the given plane as our (x, y) plane we have $w = 0$ and (y, v) independent of z. We shall find it convenient to introduce the complex variable $x + iy$ and as we shall not have occasion to use the symbol z in this section in its previous sense we shall denote $x + iy$ by z. Assuming the fluid to be *incompressible* we have, from (6), the relation

$$(25) \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

and if we further assume the motion to be irrotational we have

$$(26) \qquad \zeta \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0.$$

The equations (25) and (26) are equivalent to the statement that $u - iv$ is a function of z. We shall denote this function by $w(z)$, since there is no possibility of confusion with the third velocity component which is now everywhere zero; $w(z)$ is called the *complex velocity*. Upon introducing the function $f(z)$ defined, save for an additive constant, by the equation

$$(27) \qquad\qquad \frac{df}{dz} = w(z),$$

and denoting the real and imaginary parts of f by ϕ and ψ, respectively, we have, from (27),

$$(28) \qquad u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$

Hence ϕ is the velocity potential and the level curves $\psi =$ constant are the stream-lines. The function f is called the complex potential. The equations (28) tell us that the resolved part v_s of the velocity vector in any direction s is given by

$$(29) \qquad\qquad v_s = \frac{\partial \phi}{\partial s} = \frac{\partial \psi}{\partial \left(s + \dfrac{\pi}{2} \right)}.$$

Once the complex potential f is known the velocity components (u, v) follow upon differentiation and from these the pressure at any point may be calculated by means of (20), the motion being assumed to be steady.

8. *The Flow around a Circular Cylinder*. We shall suppose that the cylinder, of radius a and with generators perpendicular to the (x, y) plane, has a uniform velocity of translation V making an angle $-\alpha$ with the positive x axis. The flow relative to the cylinder will, therefore, have a velocity V making an angle $\pi - \alpha$ with the positive x axis at points remote from the cylindrical obstacle. In other words the complex velocity w will have the value $Ve^{i(\pi+\alpha)} = -Ve^{i\alpha}$ at $z = \infty$ so that, at $z = \infty$, f will have the form $-Ve^{i\alpha}z$ +terms whose derivatives vanish at $z = \infty$. Since the circle in which the cylinder intersects the (x, y) plane must be a portion of a stream line, ψ must be constant along it and this constant may be taken to be zero since f is undefined to the extent of an additive constant; hence the correspondence set up between points in the z plane and points of the

$f = \phi + i\psi$ plane is such that points on the circle in the z plane correspond to points on the axis of reals in the f plane. If the center of the circle in the z plane be denoted by z_0 and if $z_0 \pm ae^{i\delta}$ are two diametrically opposite points on the circle the relation

$$\frac{f - f_0}{f + f_0} = C \left(\frac{z - z_0 - ae^{i\delta}}{z - z_0 + ae^{i\delta}} \right)^2$$

will set up a correspondence of the kind desired, the ends of the segment of the real axis in the f plane being denoted by $\pm f_0$ and C being a constant. In order that f may have the proper form, $f = -Ve^{i\alpha}z + \cdots$, at ∞, it is necessary that $C = 1$ and $\delta = \pi - \alpha$, $f_0 = 2aV$. The terms in f whose derivatives vanish at $z = \infty$ may include, since f need not be uniform owing to the fact that the portion of the z plane outside the circle is not simply connected, a term of the form $A \log (z - z_0)$; for if A is a pure imaginary, the imaginary part of $A \log (z - z_0)$ is constant as we go along the circle in the z plane. If we write $A = -i\Gamma/(2\pi)$, the quantity Γ will be the circulation around any closed curve in the (x, y) plane which enlaces the cylinder once and we have

$$(30) \quad f = -V \left[(z - z_0)e^{i\alpha} + \frac{a^2 e^{-i\alpha}}{z - z_0} \right] - \frac{i\Gamma}{2\pi} \log (z - z_0).$$

We shall see in the next paragraph how Γ is determined for obstacles with a sharp trailing edge. It was precisely this determination of Γ which constituted the first significant result belonging to what we have termed the modern period.

On differentiating f it is seen that there are two points where w is zero. If $\Gamma \leqq 4\pi aV$, these two points are on the circle and are symmetrically situated with respect to the diameter of the circle which is perpendicular to the direction of flight. The two points are known as *rest points* and they are distinguished from one another by the terms *front* and *rear*. If the argument of the rear rest-point be denoted by $\pi + \beta$ we have

$$(31) \qquad \Gamma = 4\pi a V \sin (\alpha + \beta).$$

9. *Flow around an Airfoil.* If we have an obstacle in the z plane which can be mapped on a circle in an auxiliary ζ plane by means of a relationship of the type

$$(32) \qquad \zeta = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots,$$

which makes the points at ∞ in the two planes correspond to one another, it is clear from (30) that the desired complex potential is given by

$$(33) \quad f = - V \left[(\zeta - \zeta_0) e^{i\alpha} + \frac{a^2 e^{-i\alpha}}{\zeta - \zeta_0} \right] - \frac{i\Gamma}{2\pi} \log (\zeta - \zeta_0),$$

where ζ is defined by (32), a is the radius, and ζ_0 the center of the circle in the auxiliary plane. We shall be interested mainly in obstacles whose contour is of the type illustrated in Fig. 1; these have a cusp or sharp double point at the trailing edge T and so $dz/d\zeta$ will be zero there. Now the complex velocity

$$w = \frac{df}{dz} = \frac{df}{d\zeta} : \frac{dz}{d\zeta},$$

and this will be infinite at the trailing edge unless $df/d\zeta$ is zero there. This condition suffices to determine the unknown circulation Γ. Physically it amounts to saying that for a stream-lined airfoil the flow is smooth at the trailing edge. Geometrically stated, the rear rest-point on the circle in the auxiliary ζ plane must be mapped on the trailing edge of the obstacle in the z plane.

10. *The Blasius Formulas for Lift and Turning-Moment.* If ds' denote the element of arc of any curve C' enclosing the contour of the obstacle, the components of the force *per unit length* exerted on the obstacle by the fluid are, from (23),

$$F_x = \frac{\rho}{2} \oint_{C'} \{l(u^2 + v^2) - 2u(lu + mv)\}ds',$$

$$F_y = \frac{\rho}{2} \oint_{C'} \{m(u^2 + v^2) - 2v(lu + mv)\}ds',$$

where (l, m) are the (x, y) direction cosines of the outward drawn normal to C'. Since $l = dy'/ds'$ and $m = -dx'/ds'$, we readily find that

(34) $$\overline{F} \equiv F_x - iF_y = \frac{i\rho}{2} \oint_{C'} w^2 dz.$$

This important formula, due to Blasius, enables us to write down at once F_x and F_y when the development of w^2 near $z = \infty$ is known.

Similarly the turning moment M, about an axis through the origin perpendicular to the (x, y) plane, of the forces exerted by the fluid on a unit length of the cylinder, being equal to

$$\frac{\rho}{2} \oint_{C'} \{(xm - yl)(u^2 + v^2) - 2(xv - yu)(lu + mv)\}ds',$$

is seen to be the real part of

$$-\frac{\rho}{2} \oint_{C'} zw^2 dz,$$

that is,

(35) $$M = \text{R.P.} \left(-\frac{\rho}{2} \oint_{C'} zw^2 dz\right).$$

In order to apply these results we turn to (32) and (33). Thus, from

$$\zeta = z\left(1 + \frac{a_1}{z^2} + \cdots\right),$$

we find

$$\frac{1}{\zeta} = \frac{1}{z} - \frac{a_1}{z^3} + \cdots \; ; \quad \frac{1}{\zeta^2} = \frac{1}{z^2} - \frac{2a_1}{z^4} + \cdots ,$$

whence

$$\frac{df}{d\zeta} = -Ve^{i\alpha} - \frac{i\Gamma}{2\pi z} + \left(Va^2 e^{-i\alpha} - \frac{i\Gamma\zeta_0}{2\pi} \right)\frac{1}{z^2} + \cdots .$$

Multipliying this by

$$\frac{d\zeta}{dz} = 1 - \frac{a_1}{z^2} + \cdots ,$$

we find

$$w = -Ve^{i\alpha} - \frac{i\Gamma}{2\pi z} + \left(Va^2 e^{-i\alpha} - \frac{i\Gamma\zeta_0}{2\pi} + a_1 Ve^{i\alpha} \right)\frac{1}{z^2} + \cdots ,$$

$$w^2 = V^2 e^{2i\alpha} + \frac{iV\Gamma e^{i\alpha}}{\pi z}$$

$$- \left(\frac{\Gamma^2}{4\pi^2} + 2V^2 a^2 - \frac{i\Gamma\zeta_0 V e^{i\alpha}}{\pi} + 2a_1 V^2 e^{2i\alpha} \right)\frac{1}{z^2} + \cdots .$$

From this expression it follows, by (34), that

(36)
$$\overline{F} \equiv F_x - iF_y = -i\rho V\Gamma e^{i\alpha} ,$$

$$F \equiv F_x + iF_y = i\rho V\Gamma e^{-i\alpha}.$$

In other words *the force per unit length exerted by the fluid on the cylindrical obstacle has the magnitude $\pm\rho V\Gamma$ and is perpendicular to the direction of flight*, the sense of rotation from the direction of flight to the direction of the force being that of the circulation Γ. See Fig. 2. This force per unit length is called the *lift* per unit length and is denoted by L, so that we have the Kutta-Joukowski formula

(37) $$L = \rho V\Gamma.$$

If the potential function f were assumed uniform, Γ would be zero and there would be no force experienced by the obstacle; a result known as d'Alembert's paradox. The component of the force in the direction opposite to the direction of flight is called the *drag* and the drag per unit length is de-

noted by D. We see, then, that the present theory gives $D = 0$, which is not in accord with experience. It will appear in the next section that when account is taken of the fact that the length of the obstacle is not infinite this discrepancy can be removed.
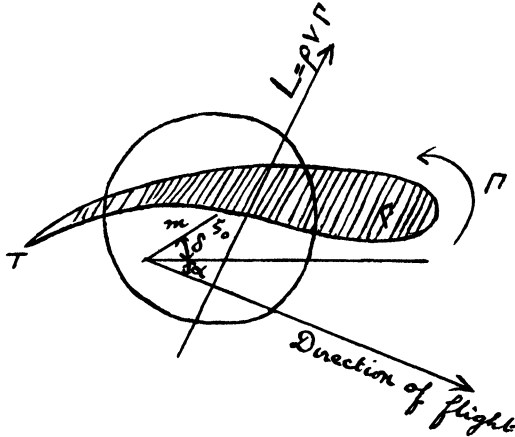

Fig. 2

If we introduce the notation $\zeta_0 = m e^{i\delta}$, $a_1 = -b^2 e^{2i\gamma}$, where m, δ, b and γ are all real, it follows that

$$M = \rho V \Gamma m \cos(\alpha + \delta) + 2\pi \rho b^2 V^2 \sin 2(\alpha + \gamma),$$

and hence the turning moment about the center $\zeta_0 = 0$ of the circle in the auxiliary ζ plane, the two planes being supposed superposed, is

(38) $M_0 = 2\pi \rho b^2 V^2 \sin 2(\alpha + \gamma).$

11. *The Circular Arc and Straight Line Profile.* The relationship $((\zeta - l)/(\zeta + l))^2 = A(z - c/2)/(z + c/2)$, where A is any constant, maps a doubly covered circular arc through the points $\pm c/2$ in the z plane on a circle through the points $\pm l$ in the ζ plane. In order that the mapping may be of the type (32) we must have $A = 1$, $l = c/4$ and the mapping is

(39) $z = \zeta + \dfrac{c^2}{16\zeta}$, $\zeta = z - \dfrac{c^2}{16z} + \cdots,$

it being necessary to take that one of the two determinations of ζ which tends to ∞ with z. If we denote by 2ϵ the angle of the circular arc, we have

$$(40) \quad \zeta_0 = il \operatorname{ctn} \epsilon = \frac{ic}{4} \operatorname{ctn} \epsilon, \qquad a = l \csc \epsilon = \frac{c}{4} \csc \epsilon,$$

$$\beta = \frac{\pi}{2} - \epsilon, \qquad b^2 = \frac{c^2}{16}, \qquad \gamma = 0,$$

so that, by (31)

$$(41) \qquad \Gamma = \pi cV \csc \epsilon \cos(\alpha - \epsilon),$$

$$(42) \quad L = \pi \rho V^2 c \csc \epsilon \cos(\alpha - \epsilon), \qquad M_0 = (\pi \rho c^2 V^2 \sin 2\alpha)/8.$$

The distance h of the lifting force $L = \rho V \Gamma$ from the center of the auxiliary circle is

$$h = M_0/L = c \sin 2\alpha \sin \epsilon / [8 \cos(\alpha - \epsilon)].$$

The particular case where $\epsilon = \pi/2$ is very important. Here we are dealing with a plane wing flying at an *angle of attack* α and we find

$$(43) \quad \Gamma = \pi cV \sin \alpha, \qquad L = \pi c \rho V^2 \sin \alpha, \qquad h = c \cos \alpha/4.$$

The number c is the length of the chord of the wing and the expression for h indicates that the point where the line of action of L meets the chord, that is, the center of pressure, does not vary with α but remains constantly one-quarter of the length of the chord from the leading edge; for ζ_0 is now the middle point of the chord. The number $c_L = L/(\frac{1}{2}\rho V^2 c)$ is called the *lift coefficient c_L* and we have the result that for a plane wing

$$(44) \qquad c_L = 2\pi \sin \alpha.$$

For small angles of attack we may put

$$(45) \qquad c_L = 2\pi \alpha.$$

so that there is a linear relation connecting c_L and α. This prediction is verified by experience for angles of attack up to about 15°; but the factor of proportionality 2π is found to be about 40% too large. The reason for this discrepancy is that the wing is not infinite in length and a corrected expression will be given in the next section.

12. *Joukowsky Profiles and their Extensions.* The relationship between z and ζ which is expressed by (39) maps the exterior of a circle through the points $\pm c/4$ in the ζ plane upon the exterior of a doubly covered circular arc, with ends at the points $\pm c/2$, in the z plane. A profile suitable for airfoils is obtained if the same mapping relationship is applied to a circle through the point $-c/4$ but *not* through the point $+c/4$ in the ζ plane. If the circle does not differ very much from the circle through the points $\pm c/4$ which maps into the doubly covered circular arc, the contour will have a rounded leading edge and a sharp trailing edge. A linear fractional substitution on a complex variable being resolvable into a translation, rotation, reflexion in a line and an inversion may be called a generalized inversion and it is evident that the mapping (39) may be written as

$$(46) \qquad z' = \zeta'^2, \quad z' = \frac{z - \dfrac{c}{2}}{z + \dfrac{c}{2}}, \quad \zeta' = \frac{\zeta - \dfrac{c}{4}}{\zeta + \dfrac{c}{4}}.$$

Now a generalized inversion sends circles into circles (straight lines being regarded as circles through the point at infinity). The generalized inversion in the ζ plane, written down in (46), sends any circle through $-c/4$ in this plane into a straight line and this is sent by the square-mapping $z' = \zeta'^2$ into a parabola in the z' plane. The profile obtained is then the generalized inverse of a parabola. Such profiles are known as Joukowsky profiles. Generalizations are obtained by writing, instead of (46),

$$(47) \qquad z' = \zeta'^n, \quad z' = \frac{z - \dfrac{c}{n}}{z + \dfrac{c}{n}}, \quad \zeta' = \frac{\zeta - \dfrac{c}{2n}}{\zeta + \dfrac{c}{2n}},$$

which is of the form (32) near $z = \infty$. This sends the exterior of a circle through the points $\pm c/(2n)$ in the ζ plane into the exterior of a region bounded by two circular arcs through the points $\pm c/n$ in the z plane, the angle between the two arcs being $(2-n)\pi$. By taking a circle in the ζ plane through $-c/(2n)$ but not exactly through $+c/(2n)$ we obtain a profile with a rounded leading edge and a sharp trailing edge; it has the crescent formed by the two circular arcs as a "skeleton" just as the Joukowsky profile has the doubly covered circular arc as skeleton. The essential difference is that the trailing edge is now a double point of the contour and not a cusp as in the case of a Joukowsky profile. The new profiles are obtained by applying a generalized inversion to the curves obtained by applying the nth power mapping $z' = \zeta'^n$ to a straight line (passing nearly through the origin) in the ζ' plane.

C. The Induced Drag of a Finite Airfoil

13. *The Formulas for the Lift and Drag.* It has been seen in the previous section that when the length of the airfoil is assumed infinite it experiences, owing to the circulation around it, a force at right angles to the direction of flight; in other words there is a lift but no drag. In order to give an explanation of this discrepancy between theory and experiment Prandtl advanced the idea that since an actual airfoil is not infinite in length the circulation around it may vary from point to point along its length. Under this hypothesis the flow loses at once its two-dimensional character but it is assumed that the results obtained for the infinite wing may be used as first approximations; the fact that the circulation is not constant along the span of the airfoil introduces a correction to this approximation. To get an

estimate of the nature and amount of this correction we shall consider the case of a plane wing whose chord c is small in comparison with its length or *span* s and we shall suppose the airfoil sufficiently represented by its trailing edge (it is then called a lifting line). If we introduce a coordinate system with its origin at the middle of the lifting line (see Fig. 3), we may draw the x axis in the direction of flight, the z axis vertically downwards and the y axis along the lifting line. The circulation around the wing in any plane perpendicular to it is assumed to be a function $\Gamma(y)$
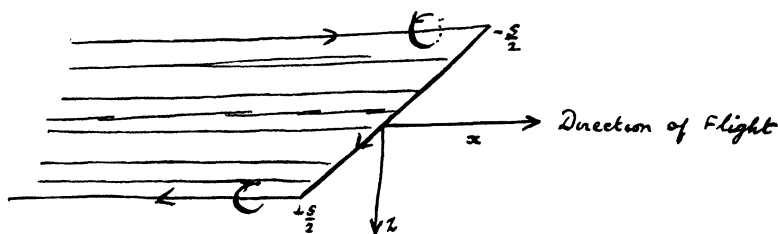

FIG. 3

of y. If then we consider the ring-shaped area bounded by two closed curves enlacing the wing, one in the plane y and the other in the plane $y+\Delta y$, there will be a flux of vorticity through this area, outwards from the wing, of amount $-(d\Gamma/dy)\Delta y$. (See (8).) We assume that this flux is concentrated at the trailing edge and is in the direction of the negative x axis, that is, is in the direction opposite to the direction of flight. Hence there will be a sheet of vortices trailing off from the trailing edge of the wing such that the strength of the vortices leaving an element Δy is $-(d\Gamma/dy)\Delta y$. Now if we have a single rectilinear vortex filament of strength μ in an indefinitely extended fluid which is at rest at infinity, the velocity distribution must be symmetrical around it and hence at a distance r from it the velocity must be of magnitude $\mu/(2\pi r)$. We *assume* that for a *semi*-infinite filament extending, as in the present case, along the negative x axis, the magnitude of the velocity at a point along the y axis distant r from the filament will be $\mu/(4\pi r)$. The direction

of the velocity will be that of the positive or negative z axis. This argument leads to the conclusion that the presence of the semi-infinite strip of trailing vortices induces at any point y of the lifting line a velocity $w(y)$ in the direction of the z axis, that is downwards, of amount

$$(48) \qquad w(y) = \frac{1}{4\pi} \int_{-s/2}^{s/2} \frac{(d\Gamma/dy)_{y=y'}}{y - y'} dy'.$$

This assumption of an induced *downwash* is the cardinal one of Prandtl's theory for the finite wing. For an infinite plane wing flying at a small angle of attack α with uniform velocity V an easy evaluation of the indeterminate form df/dz shows that the relative velocity of the wind at the trailing edge has the magnitude $V \cos \alpha$ and the direction of the negative chord of the wing and for small angles of attack α this can be regarded as of magnitude V in the direction opposite to the direction of flight. The net effect of compounding with this relative velocity V the downwash w is to rotate the relative velocity through the angle $\tan^{-1}(w/V)$ and since w is merely a small corrective term this can be equated to w/V. This rotation of the relative wind, due to the sheet of trailing vortices, may be regarded as equivalent to a rotation of the direction of flight through the same angle, that is to a diminution of the angle of attack (which is the angle from the direction of flight to the chord of the wing). The diminished angle α_e is called the *effective* angle of attack (as opposed to the *geometric* angle of attack α_g) and we have the equation

$$(49) \qquad \alpha_e = \alpha_g - \frac{w}{V}.$$

On making the assumption that we can apply the results developed for the infinite wing, once we have replaced the geometric angle of attack by the effective angle of attack, we find that the lift per unit length L' at any point y on the span of the wing is given by

$$2\pi c \cdot \frac{1}{2}\rho V^2 \cdot \left(\alpha_g - \frac{w}{V}\right),$$

and on using $L' = \rho V\Gamma$ and equation (48), we have the equation

$$(50) \qquad \Gamma(y) = \pi c V\alpha_g - \frac{c}{4} \int_{-s/2}^{s/2} \frac{(d\Gamma/dy)_{y=y'}}{y - y'} dy'$$

for the unknown function $\Gamma(y)$. For a given airfoil both c and α_g will be given functions of y; if α_g is not constant the airfoil is said to be twisted. It is apparent from (49) that even when α_g is constant α_e may vary owing to the possibility of a variable downwash $w(y)$. This variability of α_e is known as aerodynamic twisting of the airfoil.

Since the lift experienced by an infinite wing is perpendicular to the direction of flight and since the effect of the downwash is to decrease, in effect, the angle of attack by w/V, the direction of the force at any point of the wing will be rotated through an angle w/V from the direction of lift. It is this rotation of the direction of the force which introduces a component in the direction opposite to that of flight. Once $\Gamma(y)$ is determined from (50), the total lift L and drag D experienced by the wing are given by the expressions

$$(51) \qquad L = \rho V \int_{-s/2}^{s/2} \Gamma dy, \qquad D = \rho \int_{-s/2}^{s/2} w\Gamma dy,$$

(where we have identified the sine of w/V with w/V). The drag D given by (51) is known as *induced* drag, it being a matter of experience that it is too small by a certain amount depending on the shape of the airfoil and known as the profile drag.

14. *The Solution of the Integral Equation and the Determination of the Airfoil of Minimum Induced Drag having a Given Lift.* In order to obtain a numerical solution of (50) we make the substitution $y = -(s/2) \cos \theta$, so that θ runs from 0 to π along the span of the wing and imagine $\Gamma(y) = F(\theta)$ developed

in a Fourier sine series (in order that there may be no discontinuity in $\Gamma(y)$ at the wing tips, $y = \pm s/2$, we must have $\Gamma(\pm y/2) = 0$). Writing this series as $A_1 \sin \theta + A_2 \sin 2\theta + \cdots$, we see from the fact that $\Gamma(y)$ is symmetric about the middle of the wing that $A_2 = A_4 = A_6 = \cdots = 0$. In actual calculations it will be sufficient to consider the four terms $A_1 \sin \theta + A_3 \sin 3\theta + A_5 \sin 5\theta + A_7 \sin 7\theta$ in the development of $\Gamma(y)$. A simple substitution in (50) and a knowledge of the fact that

$$\int_0^\pi \frac{\cos n\phi \, d\phi}{\cos \phi - \cos \theta} = \pi \frac{\sin n\theta}{\sin \theta}$$

gives the equation

$$(52) \qquad \sum_1^\infty A_n \sin n\theta \left\{ \sin \theta + n \frac{\pi c}{2s} \right\} = \pi c V \alpha_g \sin \theta$$

for the determination of the coefficients A_n. On substituting the values $\theta = \pi/8$, $\theta = \pi/4$, $\theta = 3\pi/8$, $\theta = \pi/2$, in this and dropping the terms after A_7 we have four linear equations for the determination of the four coefficients A_1, A_3, A_5, and A_7.

The coefficients A_n being supposed known, the expressions for the lift L and the drag D in terms of them follow readily from the formulas (51). We find

$$(53) \qquad L = \rho V s A_1/4, \quad D = \frac{\pi \rho}{8} \sum_1^\infty n A_n^2 ,$$

so that the lift depends only on the first term in the Fourier development of $\Gamma(y) \equiv F(\theta)$. The downwash w is given by

$$(54) \qquad w(y) = \frac{1}{2s} \sum_1^\infty \frac{n A_n \sin n\theta}{\sin \theta} ,$$

and the drag D by

$$(55) \qquad D = \frac{\pi \rho}{8} \sum_1^\infty n A_n^2 .$$

It follows that D is a minimum, for a given value of L, when

the remaining coefficients A_3, A_5, $\cdot$ $\cdot$ $\cdot$ after A_1 in the Fourier development of $F(\theta)$, are zero. In this case the downwash has the constant value $A_1/$ $(2s)$ and there is no aerodynamic twisting of the wing. Since

$$\Gamma(y) = A_1 \sin\theta = A_1\left(1 - \frac{4y^2}{s^2}\right)^{1/2},$$

the graph of $\Gamma(y)$ against y is an ellipse and the wing is said to be elliptically loaded. Since for an infinite plane wing the lift per unit length is, for a given angle of attack, proportional to the chord c (see (43)), we may surmise that an elliptically loaded wing would be approximated to by one whose profile is approximately elliptical. A criterion of good design of an airfoil is that, for a given L, D should be as small as possible and so for a well designed airfoil we may assume the loading approximately elliptical.

15. *Comparison with Experiment.* For a finite wing having a constant chord c the ratio of the span s to the chord c is known as the aspect ratio. For a plane wing whose chord is not constant we first define the mean chord c as the quotient of the area S of the wing by its span s; then the aspect ratio A is the ratio of the span of the wing to its mean chord. Hence $A = s^2/S$. The lift coefficient c_L of the wing is defined by the equation

(56) $$c_L = L/(\tfrac{1}{2}\rho V^2 S),$$

and there is a similar equation defining the drag coefficient. For an elliptically loaded wing we have, by (53) and (56),

(57) $$c_L = \frac{\pi A}{2Vs}A_1,$$

while the downwash w has the constant value

(58) $$w = \frac{A_1}{2s} = \frac{V}{\pi A}c_L.$$

Hence the diminution in the angle of attack, induced by the

trailing vortices, is $c_L/(\pi A)$. The total lift L of the wing is

$$2\pi\alpha_e \cdot \tfrac{1}{2}\rho V^2 \cdot \int_{-s/2}^{s/2} c\,dy = 2\pi\alpha_e \cdot \tfrac{1}{2}\rho V^2 \cdot S,$$

so that $c_L = 2\pi\alpha_e$ and we have

$$(59) \qquad \alpha_g = \alpha_e + \frac{c_L}{\pi A} = \alpha_e\left(1 + \frac{2}{A}\right).$$

The aspect ratio of a wing is usually about 6, so that the lift formula for the infinite wing (using α_g) might be expected to give a result which is too large in the ratio $4:3$. This is confirmed by experience.

The drag coefficient c_D has, for an elliptically loaded wing, the value $\pi\rho A_1^2/8$ (by (55)) so that, by (57),

$$(60) \qquad\qquad c_D = c_L^2/(\pi A).$$

Hence a curve plotting D against L should be parabolic and this is verified experimentally; the actual curve obtained empirically indicates a relationship of the type

$$(61) \qquad\qquad c_D = c_{D_0} + c_L^2/(\pi A),$$

and c_{D_0} is called the coefficient of profile drag. The formulas (59) and (61) are of importance because they enable us to calculate the effect of different aspect ratios; thus for two airfoils having the same lift coefficient

$$(62) \quad \alpha_g' - \alpha_g = \frac{c_L}{\pi}\left(\frac{1}{A'} - \frac{1}{A}\right), \quad c_D' - c_D = \frac{c_L^2}{\pi}\left(\frac{1}{A'} - \frac{1}{A}\right).$$

D. Concluding Remarks

In the present address, I have tried to present in an elementary way some of those fundamental principles of modern hydrodynamical theory, which have proved useful in aeronautics, in the hope of interesting some of our hearers and readers in these questions. It would be easy to point out many unsolved problems and indeed a casual reading of the previous sections will show that even in the most elementary

problems assumptions have continually to be made the legitimacy of which is at least not proved. We append a bibliography which has been deliberately restricted to those works which we would recommend to a student wishing to orient himself in the subject and to reach research problems as soon as possible.

TEXTBOOKS

I. H. Lamb, *Hydrodynamics*. 5th ed. Cambridge University Press. The standard text on hydrodynamics.

II. H. Glauert, *The Elements of Aerofoil and Airscrew Theory*. Cambridge University Press. A fine and elementary introduction to the modern problems and methods.

III. P. Appell, *Traité de Mécanique rationnelle*, vol. 3. Gauthier Villars.

IV. L. Bairstow, *Applied Aerodynamics*. Longmans-Green.

V. R. von Mises, *Fluglehre*. Julius Springer.

These two texts are less mathematical and more physical than the other texts and are particularly valuable as complementing them.

VI. Fuchs-Hopf, *Aerodynamik*. R. C. Schmidt. This is volume 2 of a *Handbuch der Flugzeugkund*, and is very complete.

VII. A. Betz, *Tragflügel und Hydraulische Maschinen*. Handbuch der Physik, vol. 7, pp. 215–288.

VIII. Th. v. Karman and T. Levi-Civita, *Hydro- und Aerodynamik*. J. Springer.
Many papers of interest will be found in the volumes of the Zeitschrift für Angewandte Mathematik and in the publications of the National Advisory Commitee for Aeronautics. Of these latter the following may be especially recommended:

IX. N.A.C.A. Report No. 116 (Prandtl). Government Printing Office, Washington, D.C.

X. N.A.C.A. Report No. 213 (Ames). Government Printing Office, Washington, D.C.

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