$$\frac{m^3(1-m^3)^3}{(1-20m^3-8m^6)^2} = \frac{-\left(\dfrac{1+2m^3}{6m^2}\right)^3\left[1+\left(\dfrac{1+2m^3}{6m^2}\right)^3\right]^3}{\left[1+20\left(\dfrac{1+2m^3}{6m^2}\right)^3-8\left(\dfrac{1+2m^3}{6m^2}\right)^6\right]^2}.$$

Dividing out the factors $(1-20m^3-8m^6)^2(8m^3+1)^3$, to which correspond special cubics, we obtain an equation of degree 24. The 24 cubics corresponding are exactly the 24 previously mentioned and we have proved that *a necessary and sufficient condition that a non-special cubic have its (Hessian)^3 coincide with itself, is that its Hessian be projectively equivalent to itself.*

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A PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA*

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1. *Introduction.* The number of proofs given of the fundamental theorem of algebra is large. Perhaps for that very reason still another proof may not be unacceptable. The one that is offered here is not "elementary," since it makes use of some general results in analysis. Yet it may be termed simple, and may be of interest. It is our hope that this proof, which is believed to be new, may, with no great embarrassment, take its place in the family of proofs that every algebraic equation has a root.

2. *The Proof.* We consider the equation

$$(1)\qquad a_kx^k + a_{k-1}x^{k-1} + \cdots + a_1x + a_0 = 0, \quad (a_k \neq 0, k > 0).$$

There is no loss in generality in supposing that‡ $a_1 \neq 0$; and,

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‡ For suppose $a_1=0$. Make the substitution $x=y+\alpha$. We obtain an equation in y, in which the coefficient of y is a polynomial in α of degree

by dividing through by a_1, we can always make the coefficient of x unity. We then take $a_1=1$. Let us replace the constant term a_0 by $-\lambda$:

$$(2) \qquad a_k x^k + a_{k-1}x^{k-1} + \cdots + x - \lambda = 0.$$

We shall prove that (2) has a root for every λ. For $\lambda=0$ we have a root $x=0$. Let us assume that x, considered as a function of λ, can be expanded in a power series about the origin (with no constant term):

$$(3) \qquad x(\lambda) = \sum_{n=1}^{\infty} x_n \lambda^n.$$

If we substitute (3) into (2), and equate coefficients of like powers of λ, we find for $x_1, x_2, \cdots$, the series of equations

$$x_1 = 1, \quad x_2 = P_2(x_1), \quad x_3 = P_3(x_1, x_2), \cdots,$$
$$x_n = P_n(x_1, \cdots, x_{n-1}), \cdots,$$

where the P's are polynomials in their respective variables; and these permit us to determine, step by step, *and uniquely*, $x_1, x_2, \cdots, x_n, \cdots$.

The series (3) has a non-zero radius of convergence. One method of proof is to determine suitable inequalities for $|x_n|$. But it is simpler to appeal to the theory of implicit functions.

Denote the left hand member of (2) by $F(\lambda; x)$. We have $F(0; 0)=0$, $\partial F(0;0)/\partial x = 1 \neq 0$; consequently, in a sufficiently small neighborhood of $\lambda=0$ (in the λ-plane), there exists a unique analytic function $x(\lambda)$ with $x(0)=0$, which when put into (2) makes (2) an identity in λ. This function will have a convergent power series expansion, which by the uniqueness of the coefficients x_n, must coincide with (3).

For every λ inside the circle of convergence of (3), $x(\lambda)$ is a root of (2). If (3) converges for all* (finite) λ, then (2)

$k-1$, and since we know (by elementary algebra) that an equation of degree $k-1$ cannot have more than $k-1$ roots, it follows that α can be chosen so that this coefficient is not zero.

* This will be the case when $k=1$.

has a root for every λ, and the desired theorem is established. Let us then consider the contrary case, where the radius of convergence, r, is finite; and let C denote the circle of convergence. On C, $x(\lambda)$ has at least one singular point. Let $\lambda=\lambda'$ be such a singularity.

Interior to C, $x(\lambda)$ *is bounded*. For suppose the contrary. Then there will exist an infinite sequence of values λ_n, with $|\lambda_n|<r$, such that $\lim |x(\lambda_n)|=\infty$. But $|\lambda|$ being bounded, for $|x|$ sufficiently large the dominant term in the left hand member of (2) is $|a_k x^k|$, so that for n sufficiently large, $x(\lambda_n)$ cannot satisfy (2); a contradiction.

It follows that there exists an infinite sequence λ_n, with $|\lambda_n|<r$, such that $\lim \lambda_n=\lambda'$ and such that $\lim x(\lambda_n)$ *exists*.* Call this limiting value x'. Since $\lambda=\lambda_n$, $x=x(\lambda_n)$ satisfy (2), therefore x' is a root of (2) for $\lambda=\lambda'$.

We must then have $\partial F(\lambda;\, x)/\partial x=0$ for $(\lambda;\, x)=(\lambda';\, x')$. For if not, then the implicit function argument tells us that in a sufficiently small neighborhood of $\lambda=\lambda'$ there is a unique analytic function $\bar{x}(\lambda)$ with $\bar{x}(\lambda')=x'$, such that $x=\bar{x}(\lambda)$ satisfies (2) identically. This function must be an analytic continuation of $x(\lambda)$.† But this means that λ' is not a singular point of $x(\lambda)$; a contradiction.

Now the condition $\partial F(\lambda;\, x)/\partial x=0$ does not involve λ; it is in fact an algebraic equation in x of degree $k-1$. Such

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* We use Weierstrass' Theorem on an infinite set which is bounded.

† A detailed argument is as follows: By the implicit function theorem there exist two positive numbers δ, ϵ such that for every λ in the λ'-region $|\lambda'-\lambda|<\delta$ there is one and only one x in the x'-region $|x'-x|<\epsilon$ which satisfies (2) (for $\lambda=\lambda'$ the value of x being x'); and this correspondence between λ and x in the regions considered constitutes an analytic function $\bar{x}(\lambda)$. Now the λ'-region overlaps with the region of convergence of (3). Hence for n sufficiently large we shall have λ_n in the λ'-region and $x(\lambda_n)$ in the x'-region, and this requires that $x(\lambda_n)=\bar{x}(\lambda_n)$. Now about this point $\lambda=\lambda_n$ (and for the value $x=x(\lambda_n)$), we can reapply the implicit function theorem. In a sufficiently small neighborhood of $\lambda=\lambda_n$ we find a unique analytic function of λ taking on the value $x(\lambda_n)$ at λ_n; and this function of course coincides with $\bar{x}(\lambda)$. But $x(\lambda)$ is analytic in a sufficiently small neighborhood of λ_n, and has there the value $x(\lambda_n)$, so that we must have $x(\lambda)\equiv\bar{x}(\lambda)$ in the neighborhood of λ_n.

an equation cannot be satisfied by more than $k-1$ values of x, so that (as we see from (2)) there cannot be more than $k-1$ singular points λ on C. We may then by analytic continuation follow the function $x(\lambda)$ across C; and for every λ in the circle of convergence of each continuation, $x(\lambda)$ will, by the principle of permanence, be a root of (2).

The preceding arguments apply to these new circles of convergence, so that we can extend the function $x(\lambda)$ throughout the entire (finite) λ-plane, with the exception of a *finite number* of singular points. And at each non-singular point λ, $x(\lambda)$ is a root of (2). But also at each singular point we have a root.* The theorem is thus established.

3. *Remark*. A previous argument can be used to show that in every bounded λ-region, $x(\lambda)$ is bounded. This is true in particular in the neighborhood of the singularities of $x(\lambda)$. In consequence, *the singularities*† *of* $x(\lambda)$ *are branch points at which the function remains finite.* For, the only other possibilities are that a singularity is either a pole or an essential singularity (whether or not also a branch point); but in either case there would exist an infinite sequence of points λ approaching the singularity, for which $|x(\lambda)|$ would approach infinity, and this contradicts the boundedness of $x(\lambda)$.

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* The argument used for $\lambda=\lambda'$ on C applies.

† That is, in the finite plane.