CUBICS WHOSE (HESSIAN)^n'S ARE THEMSELVES*

BY F. E. WOOD

The four equianharmonic, four degenerate and six harmonic cubics which belong to the syzygetic pencil $\lambda C+\mu H=0$, where $C=0$ is a general cubic and $H=0$ is its Hessian, have been studied in much detail. We denote them as the 14 *special* cubics of the pencil. It is known that the Hessian of a degenerate or equianharmonic cubic is a degenerate cubic, and that the (Hessian)^2, that is, the Hessian of the Hessian—of a harmonic cubic is itself and that the harmonic cubics are characterized by this property. The more general problem has been considered by Hostinský only, who entirely by calculation has obtained certain results for $n=3$ and $n=4$.† In this note a characterizing property for $n=3$ is obtained and certain general results noted, the method being quite different. The theorems of this note are believed to be new, though a part of Theorem 2 could be obtained from the work of Hostinský.

Denote by γ_n a cubic whose (Hessian)^n is itself and whose (Hessian)^t for every value $t<n$ is not itself. Let m denote the parameter of a cubic, and $m^{(n)}$ the parameter of its (Hessian)^n; then

$$m^{(r+1)} = -\frac{1+2(m^{(r)})^3}{6(m^{(r)})^2}, \qquad (r=0,1,\cdots,n-1),$$

where $m^{(0)}=m$. Then $m^{(n)}=m$ gives an equation in m of degree 3^n, which is also satisfied by any cubic whose (Hessian)^r is itself, for r a divisor of n. Dividing out from the equation the factors corresponding to these cubics, one will have in general an equation whose roots give cubics γ.

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† B. Hostinský, *Sur les Hessiennes successives d'une courbe du troisième degré*, Proceedings of the International Congress, Cambridge, 1912, vol. 2, pp. 102–104.

None of the cubics involved can be special, when $n>2$. It can be shown that to every non-special cubic of a syzygetic pencil there are eleven others forming a group of twelve, each projectively equivalent to the others. The Hessians of such a group form another group of twelve projectively equivalent cubics, all distinct unless the group of Hessians is a group of special cubics; in which case for no value of n will the (Hessian)^n of one of the original group be itself.

If the group of Hessians is each time different from all the preceding groups, then the existence of one non-special cubic γ_n implies the existence of $12n$ such cubics. For every value of n the degenerate cubics will appear, and when n is a prime and greater than 2 the parameters corresponding to the degenerate cubics will be the only ones to appear in the equation, mentioned above, of degree 3^n. And one notes that 3^n-3, n a prime, is divisible by $12n$ when $n>3$. Thus *the totality of cubics γ_n form k groups of* 12 *projectively equivalent cubics each. When n is a prime and greater than* 3, *k is a multiple of n.*

Consider $n=3$. Let a_1 be a non-special cubic whose (Hessian)^3 is itself; let $a_2, \cdots, a_{12}$ denote the other cubics of the syzygetic pencil projectively equivalent to a_1, forming all together the group A; denote the Hessian of a_i by b_i, the group of b_i by B. Then there are two possible cases; either the group B is not, or is, the group A. In the first case one finds that there must be 36 cubics γ_3, which is impossible since for $n=3$, $3^n-3=24$. In the second case one finds that the continued Hessians of A form the group A, the twelve cubics of A being arranged in four groups of three, such that the continued Hessians of the cubics of any one triple are the triple itself.

Conversely, consider the non-special cubics which have the property that the Hessian of one such is projectively equivalent to it. Let m be the parameter of the cubic, then $-(1+2m^3)/(6m^2)$ is the parameter of its Hessian; hence we have the equation

$$\frac{m^3(1 - m^3)^3}{(1 - 20m^3 - 8m^6)^2} = \frac{-\left(\frac{1 + 2m^3}{6m^2}\right)^3\left[1 + \left(\frac{1 + 2m^3}{6m^2}\right)^3\right]^3}{\left[1+20\left(\frac{1 + 2m^3}{6m^2}\right)^3 - 8\left(\frac{1 + 2m^3}{6m^2}\right)^6\right]^2}.$$

Dividing out the factors $(1-20m^3-8m^6)^2(8m^3+1)^3$, to which correspond special cubics, we obtain an equation of degree 24. The 24 cubics corresponding are exactly the 24 previously mentioned and we have proved that *a necessary and sufficient condition that a non-special cubic have its* (*Hessian*)^3 *coincide with itself, is that its Hessian be projectively equivalent to itself.*

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A PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA*

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1. *Introduction.* The number of proofs given of the fundamental theorem of algebra is large. Perhaps for that very reason still another proof may not be unacceptable. The one that is offered here is not "elementary," since it makes use of some general results in analysis. Yet it may be termed simple, and may be of interest. It is our hope that this proof, which is believed to be new, may, with no great embarrassment, take its place in the family of proofs that every algebraic equation has a root.

2. *The Proof.* We consider the equation

$$(1) \quad a_kx^k + a_{k-1}x^{k-1} + \cdots + a_1x + a_0 = 0, \quad (a_k \neq 0, k > 0).$$

There is no loss in generality in supposing that‡ $a_1 \neq 0$; and,

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‡ For suppose $a_1 = 0$. Make the substitution $x = y + \alpha$. We obtain an equation in y, in which the coefficient of y is a polynomial in α of degree