CONCERNING R. L. MOORE'S AXIOMS Σ_1 FOR PLANE ANALYSIS SITUS*

BY R. L. WILDER

1. *Introduction.* R. L. Moore has proposed† a system, Σ_1, of eight axioms for plane analysis situs. That a space S satisfying this system is in one-to-one continuous correspondence with two-dimensional euclidean space was shown by Moore in a later paper.‡

It is the purpose of the present paper to show that the set Σ_1 may be reduced to a set of *seven* Axioms, by the elimination of Axiom 6, which is a consequence of the other Axioms. Doubt as to the independence of Axiom 6 was raised in the mind of the author by noticing that the independence examples given for Axioms 6 and 7 on pp. 162 and 163 of F.A. are not valid, and by the subsequent finding of an independence example for Axiom 7 accompanied by failure to find any independence example for Axiom 6.

2. *Independence of Axiom* 7. The independence of Axiom 7 is established be the following example:

Let the space considered by ordinary euclidean space of two dimensions. Choose a pair of rectangular axes OX and OY. For every positive integer n, let points be defined as follows: $A_n=(0,-1/n)$, $B_n=(1/n,-1/n)$, $C_n=(1/n, 1/n)$, $D_n=(0, 1/n)$, $E_n=(0, 0)$, $F_n=(1/(2n), 0)$. Let T_n be the bounded domain whose boundary is the rectangle $A_nB_nC_nD_n$ together with the straight line interval E_nF_n. Then a point set R is a region if and only if R is either the interior of some simple closed curve or identical with some T_n. That the

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† *On the foundations of plane analysis situs*, Transactions of this Society, vol. 17 (1916), pp. 131–164. Referred to hereafter as F. A.

‡ *Concerning a set of postulates for plane analysis situs*, Transactions of this Society, vol. 20 (1919), pp. 169–178.

conditions of Axiom 7 are violated at E_n for every T_n is easily evident.

3. *Proof of Axiom* 6 *on the Basis of Axioms* 1–5, 7, 8. In the proof of Theorems 1–27, inclusive, of F.A., no use is made of Axiom 6. The proof given of Theorem 28 is based on both Axioms 6 and 7, but can be made independent of Axiom 6 as follows:

Proof of Theorem 28 *without use of Axiom* 6: Let the proof for the case where M denotes the exterior of J be given as in F.A. Denote the interior of J by R.

Let $\overline{R}$ be any region about O, and let K be a region about O which lies wholly interior to $\overline{R}$ and fails to contain some point, D, of J. Then by the preceding part of the argument there exists an arc AXB such that (1) A and B are on J, (2) AXB, except for its end points, is common to M and K, (3) of the two arcs into which A and B divide J, that one which contains O lies in K.

Clearly that arc of J from A to B which contains D has only A and B in common with AOB. By Theorem 27 of F.A., either D is without $AXBOA$ or O is without $AXBDA$. But since $AXBOA$ is in K, the latter case would imply that D is a point of K (Theorem 21), so that the former case must hold. Then by the second part of Theorem 27, the interior of $AXBDA = AOB$ (except for its end points) + the interior of $AOBDA$ + the interior of $AXBOA$.

Let T be a region about O and lying wholly interior to $AXBDA$ and K. Then by the preceding part of the proof, there exist two points A' and B' on $AXBOA$ and an arc $A'X'B'$ such that $A'X'B'$ except for its end points is interior to T and exterior to $AXBOA$, and that arc of $AXBOA$ from A' to B' which contains O lies in T. As the only points of $AXBOA$ that lie in T are on AOB, it is clear that the arc $A'OB'$ is a subset of AOB and hence of J. That $A'X'B'$, except for its end points, is a subset of R is evident at once since it lies interior to $AXBDA$ but contains no point of AOB or the interior of $AXBOA$.

Since T is a subset of K, which in turn is a subset of $\overline{R}$, we see that $A'X'B'$ is an arc such that (1) A' and B' are on J, (2) $A'X'B'$ is common to $\overline{R}$ and R, (3) of the two arcs into which A' and B' divide J that one which contains O lies in $\overline{R}$.

In the proof of the remainder of the theorems of F.A., that is, Nos. 29–52, inclusive, no use is now made of Axiom 6.

I shall now prove the following theorem without use of Axiom 6:*

THEOREM A. *If R is a region and J is the boundary of R, then J is a simple closed curve.*

Two proofs will be given. The first of these is based entirely on F.A. and may be understood by a reader who has no further familiarity with the existing literature of analysis situs. The second proof has the advantage of being quite brief, and to one who is familiar with the various results that have been obtained in later papers dealing with plane analysis situs—especially with the theory of continuous curves—it is by far preferable.

FIRST PROOF. This proof is based on Theorem 48 of F.A. That J satisfies all conditions† of this theorem except (3) is readily apparent if we let $S_1 \equiv R$, and $S_2 \equiv S - R'$. I shall proceed to show that condition (3) is satisfied.

Let x be any point of J, and let P be a point of R. Let $J_1, J_2, J_3, \cdots$ be a sequence of simple closed curves every one of which encloses x and such that (1) for every positive integer i, J_{i+1} lies interior to J_i, and (2) x is the only point common to the regions bounded by the simple closed curves of this sequence. (Theorems 5 and 36 of F.A.)

As x is a limit point of R, there exists in R a sequence of distinct points, $P_1, P_2, P_3, \cdots$, having x as a sequential limit point and such that for every positive integer i, P_i is interior to J_i. By Theorem 16, P and P_1 are the extremities

* See Theorem A, p. 163 of F. A.

† That the first of these conditions is superfluous has been shown by P. M. Swingle. See this Bulletin, vol. 34 (1928), pp. 607–618. This theorem (No. 48 of F. A.) was first stated by Schoenfliess.

of an arc a_1, and for every positive integer n, P_n and P_{n+1} are the extremities of an arc t_{n+1}, such that the arcs a_1, t_2, t_3, t_4, $\cdots$ all lie in R. Let x_2 be the last point of a_1 on t_2 in the order from P_1 to P_2. That portion of t_2 from x_2 to P_2 is an arc a_2. Let x_3 be the last point of a_1+a_2 on t_3 in the order from P_2 to P_3. That portion of t_3 from x_3 to P_3 is an arc a_3. Continuing in this way indefinitely, there is obtained a sequence of arcs a_1, a_2, a_3, $\cdots$, such that for every positive integer n, a_n has only one point, that is an end point x_n, in common with the set $a_1+a_2+\cdots+a_{n-1}$.

(1) If for every value of n there exists a positive integer k, such that a_k is the last arc of the sequence of arcs $\{a_i\}$ having points on J_n, then it is easily shown that the set $\sum_1^\infty a_n$ together with the point x contains an arc from P to x which lies, except for x, wholly in R.

(2) Suppose there exists a positive integer n such that infinitely many of the arcs of the sequence $\{a_i\}$ have points in common with J_n. Then infinitely many arcs a_i of the sequence have points exterior to J_{n+1} as well as P_i interior to J_{n+2}. For every such arc, a_i, let $\overline{A}_i$ be the first point of J_{n+1} on a_i in the order from P_i to x_i (letting x_1 denote P). Then let $\overline{B}_i$ be the first point of J_{n+2} on that portion of a_i from $\overline{A}_i$ to P_i, in the order from $\overline{A}_i$ to P_i. From $\overline{A}_i$ to B_i there exists an arc $\overline{A}_i\overline{B}_i$ which is a subset of a_i, and such that if I_{n+1} is the set of all points between J_{n+1} and J_{n+2} (see footnote, p. 157 of F.A.), $\overline{A}_i\overline{B}_i$, except for its end points, is a subset of I_{n+1}. Call the set of all such arcs $\{a_i^*\}$. It is easily shown that no two arcs of the set $\{a_i^*\}$ have any points in common.† Since J_{n+1} is bounded and closed, the infinite set of points of type $\overline{A}_i$ has at least one limit point A on J_{n+1} (Theorem 13). If z is any other point on J_{n+1} then at least one of the arcs into which A and z divide J_{n+1} must contain an infinite set of points of type $\overline{A}_i$ having A as a limit point; call this arc Az. Then

† See pp. 344–345 of my paper *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), pp. 341–377.

from the points of type $\overline{A}_i$ can be selected an infinite sequence $A_1, A_2, A_3, \cdots$, in the order from z to A on Az, and having A as a sequential limit point. The set of points $B_1, B_2, B_3, \cdots$, where B_n is the other end point of the arc of $\{a_i^*\}$ to which A_n belongs has a sequential limit point B. There is obtained thus a sequence $A_1B_1, A_2B_2, A_3B_3, \cdots$, of arcs of the set $\{a_i^*\}$ arranged in a definite order. Call this sequence the set $\{A_iB_i\}$.

The set $\{A_iB_i\}$ has a limiting set, M_1, which† evidently contains A and B. Let w be a point of M_1 in I_{n+1} (that such a point exists is easily shown; for instance, any simple closed curve which encloses J_{n+2} and lies within J_{n+1} will contain such a point), and let C_1 be a simple closed curve enclosing w and lying wholly in I_{n+1} but not enclosing J_{n+2}. Let A_kB_k be an arc of the set $\{A_iB_i\}$ which has points interior to C_1, and let y be one such point. Let s be an arc from y to w lying wholly interior to C_1, and let m be the first point of M_1 on s in the order from y to w. Denote that portion of s from y to m by t. From the order of the arcs $\{A_iB_i\}$ it follows that there exists a positive integer j such that every arc A_iB_i for which $i>j$ has a point y_i on t, and such that m is the sequential limit point of the sequence $y_j, y_{j+1}, y_{j+2}, \cdots$.

(a) If m is a point of R, there exists a region R_1 which contains m and lies wholly in R. Then all but a finite number of the arcs of the set $\{A_iB_i\}$ are joined to one another by a sub-arc of t which lies wholly in R.

(b) If m is not a point of R, then it must be a point of J. By Axiom 7 there exist in C_1 two regions K and $\overline{K}$ such that $\overline{K}$ contains m, K lies in $S-R'$, and all those points of J that lie in $\overline{K}$ are points also of the boundary of K.

For each i, A_iB_i and $A_{i+1}B_{i+1}$, together with those arcs A_iA_{i+1} and B_iB_{i+1} of J_{n+1} and J_{n+2}, respectively, that contain no end points of the set $\{A_iB_i\}$, form a simple closed curve K_i which cannot enclose m. For every positive integer

† A point x is contained in M_1 if it is a sequential limit point of a sequence of points $x_1, x_2, x_3, \cdots$, where for every n, x_n is a point of A_nB_n.

$r \geqq k$, there is an arc b_r, subset of t, which has one end point on each of the arcs A_rB_r and $A_{r+1}B_{r+2}$, and which lies, except for these end points, entirely within K_r. For each r there is at most a finite number of such arcs b_r.

If only a finite number of arcs of type b_r contain points of $S-R$, then all but a finite number of the arcs $\{A_iB_i\}$ are joined by arcs of R which lie wholly interior to C_1. That this is indeed the case will follow if it can be shown that there cannot be infinitely many of the arcs of type b_r that contain points of $S-R$. This is done as follows: If an arc b_r contains a point of $S-R$, then, since its end points are in R, it must contain at least one point, P_r, of J. Let one such point be selected for each simple closed curve K_r $(r \geqq k)$. The point m is the sequential limit point of the set of points $\{P_r\}$, provided infinitely many such points exist. Then, since $\overline{K}$ contains m, $\overline{K}$ will contain at least two distinct points P_c and P_d, of type P_r, and by the conditions of Axiom 7 these points are on the boundary of K. If I_c and I_d denote, respectively, the interiors of K_c and K_d, then I_c and I_d have no point in common since no two points of type P_r lie interior to the same curve of type K_r. But then K must contain points in both I_c and I_d, and being a connected set must also contain a point, Q, on K_c. That Q is not on J_{n+1} or J_{n+2} is evident, since K is a subset of C_1. Hence Q is a point of some arc of the set $\{A_iB_i\}$. But this is impossible, since K lies in $S-R'$, and the arcs $\{A_iB_i\}$ all lie in R. Thus there cannot exist infinitely many points of type P_r, and the conclusion stated in the first sentence of this paragraph holds in any case.

There must, then, exist a number N_1, such that for $u > N_1$ and $v > N_1$, A_uB_u and A_vB_v are joined by an arc of R which lies wholly interior to C_1. A fortiori, those arcs of the set $\{a_i\}$ of which the arcs of the set $\{A_iB_i\}$ of subscript $> N_1$ are subsets, are joined by arcs of R in the same way. Call the set of such arcs S_1.

For each arc a_i of S_1 that has points interior to J_{n+3} let $\overline{E}_i$ be the last point of J_{n+2} on a_i in the order from x_i to P_i, and let $\overline{F}_i$ be the first point of J_{n+3} on that portion of a_i from $\overline{E}_i$ to P_i, in the order from $\overline{E}_i$ to P_i. Each such arc a_i contains an arc $\overline{E}_i\overline{F}_i$, such that $\overline{E}_i\overline{F}_i$, except for its end points, lies wholly in I_{n+2} (where I_{n+2} is the set of all points between J_{n+2} and J_{n+3}). It can be shown, by the methods employed above, that from the arcs of the latter collection can be selected an infinite sequence, E_1F_1, E_2F_2, E_3F_3, $\cdots$, having the property that there exists a number N_2 such that for $u>N_2$ and $v>N_2$, E_uF_u is joined to E_vF_v by an arc of R which lies wholly within I_{n+2}. The set of all those arcs of S_1 which contain arcs of this sequence of subscript $>N_2$ denote by S_2.

Continuing in this way it is shown that there exists an infinite sequence of sets S_1, S_2, S_3, $\cdots$, of arcs of $\{a_i\}$, such that for any positive integer j (1) S_{j+1} is a subset of S_j, (2) if a_u and a_v are any two arcs of $\{a_i\}$ which belong to S_j, there is an arc ab of R such that (i) one end point of this arc, a, is a point of a_u, and the other end point, b, is a point of a_v, (ii) the arc ab lies wholly interior to I_{n+j} (the set of all points between J_{n+j} and J_{n+j+1}), and (iii) the arcs aP_u and bP_v, subsets of a_u and a_v, respectively, lie wholly interior to J_{n+j}.

The first arc of the sequence $\{a_i\}$ which belongs to S_1 denote by f_1. The first arc of $\{a_i\}$ after f_1 which belongs to S_2 denote by f_2. Then there exists, in I_{n+1}, an arc of R whose end points are a point x' of f_1 and a point y' of f_2, and such that the arc $y'P_e$, a subset of the arc $f_2\equiv a_e$, lies wholly interior to J_{n+1}. The first arc of $\{a_i\}$ after f_2 which belongs to S_3 denote by f_3. Then there exists, interior to I_{n+2}, an arc of R having as end points a point x'' of $y'P_e$ and a point y'' of f_3, and such that the arc $y''P_h$, subset of $f_3\equiv a_h$, lies wholly interior to J_{n+2}. Continue this process indefinitely. Then, if $f_1\equiv a_q$, the set $a_1+a_2+\cdots+a_q+x'y'+y'P_e+x''y''+y''P_h+\cdots+x$ can be shown to contain an arc from P to x which lies, except for x, wholly in R.

To show that if P is exterior to R, then there exists an arc from P to x which lies, except for x, wholly in $S-R'$, we can proceed as above, with only slight modification of the proof.

It follows that condition (3) of Theorem 48 of F.A. is satisfied, and therefore J is a simple closed curve.

SECOND PROOF.* By virtue of the results obtained in R. L. Moore's paper *Concerning a set of postulates for plane analysis situs*,† space S is equivalent to a euclidean plane. We may, then, make free use of theorems established for the euclidean plane. By the Phragmen-Brouwer theorem, since J is the common boundary of the domains R (bounded) and $S-R'$ it is a bounded continuum. Suppose it is not a continuous curve. Then there exist two concentric circles k_1 and k_2 and a countable infinity of mutually exclusive continua $\overline{M}$, M_1, M_2, $\cdots$, satisfying all the conditions of the theorem of section 3 of R. L. Moore's *Report on continuous curves from the viewpoint of analysis situs*.‡ Let P denote a point of $\overline{M}$ lying in the point set $\overline{H}$ composed of the set of all points between k_1 and k_2 and let $\overline{R}$ denote a region containing P and lying, together with its boundary, in $\overline{H}$. By Axiom 7 there exist, in $\overline{R}$, two regions K and $\overline{K}$ such that $\overline{K}$ contains P, K lies in $S-R'$ and all points of the boundary of R that lie in $\overline{K}$ belong to the boundary of K. There exist four continua of the sequence M_1, M_2, $\cdots$, that contain points of $\overline{K}$. They may be so labelled x, y, z, w that if a connected subset of $\overline{H}$ contains a point of x and a point of z then it contains a point of $y+w$. The boundary of K contains a point A of x and a point B of z. The point set $K+A+B$ is a connected subset of $\overline{H}$. Hence K contains a point of $y+w$. But $y+w$ and K are subsets of J and of $S-J$ respectively. Thus the supposition that J is not a continuous curve leads to a contradiction. Furthermore, by Theorem 20 of F.A., J is the outer boundary of R. Hence J is§ a simple closed curve.

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* This proof was suggested to the author by R. L. Moore, to whom the original manuscript of this paper was sent for criticism prior to its being offered for publication.

† Loc. cit.

‡ This Bulletin, vol. 29 (1923), pp. 296–297.

§ R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), p. 258, Theorem 4.

THEOREM B. (Axiom 6 of F.A.) *If R and $\bar{R}$ are regions and P is a point in $\bar{R}$ and on the boundary of R, then there exist in $\bar{R}$ two regions K and $\bar{K}$ such that $\bar{K}$ contains P, K lies in R and all those points of the boundary of R that lie in $\bar{K}$ are points also of the boundary of K.*

PROOF. By Theorem A the boundary of R is a simple closed curve J. By Theorem 28 of F.A. there exists a simple continuous arc AXB such that (1) A and B are on J, (2) AXB, except for its end points, is common to R and $\bar{R}$, (3) of the two arcs into which A and B divide J, that one, APB, which contains P lies in $\bar{R}$. Let the interior of the simple closed curve formed by AXB and APB be denoted by K, and let $\bar{K}$ be any region which contains P, lies wholly in $\bar{R}$, and contains no point of AXB nor of that arc of J which does not contain P. The regions K and $\bar{K}$ defined in this manner satisfy the conditions of the theorem.

In conclusion, it is perhaps interesting to note that in the above proof of Theorem A we have also proved the following theorem.

THEOREM C. *If D is a bounded domain (as defined in F.A., p. 136) such that (1) $S-D'$ is connected, and (2) if P is a boundary point of D and R is a region containing P then there exist in R two regions K and $\bar{K}$ such that $\bar{K}$ contains P, K lies in $S-D'$ and all those points of the boundary of D that lie in $\bar{K}$ are points also of the boundary of K, then the boundary of D is a simple closed curve.*

Furthermore, by virtue of a theorem due to J. R. Kline* we can state the following theorem.

THEOREM D. *If D is an unbounded domain satisfying conditions (1) and (2) of Theorem C, then the boundary of D is an open curve.*

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* *The converse of the theorem concerning the division of a plane by an open curve*, Transactions of this Society, vol. 18 (1917), pp. 177–184. See in this connection the article of P. M. Swingle referred to in a preceding footnote.